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ABSTRACT

This guide was written under the basic assumptions that the mathematics curriculum must continuously change and that mathematics is most effectively learned through a spiral approach. Further, it is assumed that the audience will be members of district mathematics curriculum committees. Instructional objectives have been organized to reveal the continuity of development of a strand from kindergarten through grade 12. The guide was developed to serve four purposes: (1) provide a model and resource to local committees; (2) suggest criteria for content selection and strategies for curriculum articulation, implementation, and evaluation; (3) provide a description of the K-12 mathematics program for pre-service teachers; and (4) serve as a basis for all mathematics-related activity at the Department of Public Instruction in Wisconsin. The guide is organized into 12 sections. Sections 2-8 cover goals and objectives of specific content areas organized into seven strands: (1) number/numeration; (2) measurement; (3) geometry; (4) statistics; (5) algebra; (6) discrete mathematics; and (7) problem solving. Objectives are grouped by grade level. Section 9 considers college preparatory programs. Section 10 describes the curriculum development process. Section 11 describes issues affecting curriculum selection and instruction. Section 12 lists resources. The appendices, which comprise one-third of the guide, provide three categories of materials: (1) position papers and future-oriented papers by such organizations as the National Council of Supervisors of Mathematics the National Council of Teachers of Mathematics (NCTM) and the National Science Board Commission on Precollege Education in Mathematics, Science, and Technology; (2) outlines of learning objectives in mathematics for college-bound students (by the College Board and NCTM) and in Wisconsin for college-bound and non-college-bound students, as well as for Grades 3, 7 , and 10; and (3) a federally-funded report, "Girls, Math, and Science," published by the Center for Adolescence, University of North Carolina. (JM)

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A Guide to Curriculum Planning
in
Mathematics

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Foreword

The first curriculum guide in mathematics was published in 1967. At that time the state superintendent observed, "In a dynamic society such as ours, change is constant." That statement is no less true today. A changing society requires a changing curriculum, a curriculum which retains the essential components of the old while incorporating the emphases appropriate to the needs of citizens in the new society.

Elementary and secondary mathematics programs have two fundamental goals. They must provide the mathematics instruction necessary for young people to become effective citizens. For a smaller number of students mathematics programs must provide opportunities for advanced training in post-secondary institutions in preparation for careers requiring advanced mathematics, science, and engineering. School mathematics programs must seek an appropriate balance between these two goals.

The Mathematics Curriculum Task Force has rejected the notion that problems with school mathematics programs can be eliminated by teaching rigorous mathematics and setting very high standards. While these methods may be appropriate for some students under some circumstances, they are not appropriate for all students. In fact, many failures in teaching mathematics can be attributed to our inability to make mathematics concrete and meaningful to the large numbers of students who are not destined to become mathematicians.

This guide will serve as the focus of Department of Public Instruction mathematics activity during the next five years. In addition to promoting the curriculum advocated by this guide, department personnel will work to assure that other programs such as the competency testing program, the assessment program, teacher certification programs, continuing professional development programs, and high school graduation requirements are all fully compatible with the recommendations of the guide.

The recommendations are not intended to be a radical departure from present practice. Evolution, not revolution, is the intent. Districts are encouraged to create an evolutionary plan for implementation by making gradual changes over the next five years. The intent of this guide is to provide a better mathematics curriculum for all students. I know that all districts are committed to this goal and hope that this guide serves as a valuable aid to district mathematics committees as they continue in this important work.

Herbert J. Grover
State Superintendent

Acknowledgments

To identify the instructional objectives which constitute an exemplary mathematics program, the state superintendent appointed the Wisconsin Mathematics Curriculum Task Force, consisting of leaders in mathematics education from throughout Wisconsin. Members of the task force are thoroughly acquainted with all major national reports on mathematics education produced within the last 10 years, and are also thoroughly acquainted with the realities of instruction in elementary and secondary classrooms. Each task force member sought to make appropriate compromises between the ideal advocated in the national reports, and the reality of mathematics instruction.

Over the past two years, the task force members have generously given their time and expertise to preparing, discussing, and revising drafts of this guide. The philosophy expressed in section 1 of the guide and the instructional objectives identified in sections 2 through 8 represent the consensus of the task force. It should not be assumed, however, that every task force member endorses every recommendation. The content editor retains final responsibility for the content of the guide.

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1

National Reports

. . . conditions which led to the development of the curriculum in the eighteenth, nineteenth, and twentieth centuries have changed.

Within the last decade a number of national reports have addressed the mathematics curriculum taught in both elementary and secondary schools. These reports have focused on identifying the curriculum as it is and as it should be, noting that conditions which led to the development of the curriculum in the eighteenth, nineteenth, and twentieth centuries have changed. Mathematics has changed; knowledge about how students learn has also changed.

The Wisconsin Mathematics Curriculum Task Force used the reports mentioned below as a basis for its recommendations. The reports, or relevant portions of the reports, are included in the guide's appendixes. Curriculum planners are encouraged to review these documents and keep them in mind when making curriculum decisions.

In *Overview and Analysis of School Mathematics Grades K-12* (1975), the National Advisory Committee on Mathematical Education provided an account of curriculum reform during the period from 1955 to 1975. They described current programs and issues, patterns of instruction, teacher education, and evaluation, concluding with recommendations on policy (including curriculum content) and research. The content recommendations stress the applications of mathematics and the incorporation of calculators and computers in the mathematics program (see Appendix A).

In 1977 the National Council of Supervisors of Mathematics (NCSM) adopted its *Position Paper on Basic Mathematical Skills* which included the identification of ten basic skill areas (see Appendix B). All serious studies of mathematics curriculum published since the NCSM statement refer to its recommendations and support them. The National Council of Teachers of Mathematics, the Association of State Supervisors of Mathematics, and the Wisconsin Mathematics Council have endorsed the report.

In 1980 the National Council of Teachers of Mathematics (NCTM) released *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (see Appendix C). That report makes extensive recommendations regarding the role of problem solving, calculators, and computers. It also advocates three years of mathematics as a requirement for all students in grades 9-12.

The Conference Board of the Mathematical Sciences, in response to a request from the National Science Board Commission on Precollege Education in Mathematics, Science and Technology, met in 1982 to discuss the mathematics curriculum. Their report, *The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not* (see Appendix D), identifies topics from the curriculum which should receive more emphasis and topics which should receive less emphasis. That report was included as an appendix to the National Science Board Commission on Precollege Education in Mathematics, Science and Technology report, *Educating Americans for the 21st Century*, a plan of action for improving mathematics, science and technology education for all American elementary and secondary students so that their achievement is the best in the world by 1995. A portion of this report, "Improving What Is Taught and Learned," is included as Appendix E.

In 1983 many national studies or recommendations for the improvement of education appeared. The National Commission on Excellence in Education produced *A Nation At Risk: The Imperative for Educational Reform*. Like the National Council of Teachers of Mathematics and the National Science Board Commission on Precollege Education in Mathematics, Science and Technology, the National Commission on Excellence in Education called for a minimum of three years of mathematics in high school.

The commission stressed all high school graduates should understand geometric and algebraic concepts, elementary probability and statistics, applications of mathematics in everyday situations, estimation and approximation, and measurement. It advocated:

> In addition to the traditional sequence of studies available for college-bound students, new, equally demanding mathematics curricula need to be developed for those who do not plan to continue their formal education immediately.

. . . all high school graduates should understand geometric and algebraic concepts, elementary probability and statistics, applications of mathematics in everyday situations, estimation and approximation, and measurement.

In December, 1983, a conference of national leaders in mathematics education was held in Madison. The purpose was "to review the evidence and recommendations favoring reform of school mathematics that have been made by many commissions and groups during the past two years . . . [and] to propose next steps that could be taken to advance the improvement process." The report, *School Mathematics: Options for the 1990s* (see Appendix F) contains ten recommendations for improving elementary and secondary mathematics education.

The National Council of Teachers of Mathematics in March, 1984, sponsored a conference to consider "The Impact of Computing Technology on School Mathematics." That conference resulted in a report by the same title (see Appendix G).

The College Board in 1983 produced its report *Academic Preparation for College: What Students Need To Know And Be Able To Do* (see Appendix H).

In 1984 the National Council of Teachers of Mathematics published *Computing & Mathematics: The Impact on Secondary School Curricula*. This publication is the report on a conference held at the University of Maryland in 1982. It presents "an unequivocal dissent from the spirit and substance of efforts to improve school mathematics that seek broad agreement on conservative curricula." Curriculum planning committees are urged to acquaint themselves with that report.

The impact is not restricted to using new tools to do the same mathematics. Computer technology can be expected to "shift the focus of instruction from an emphasis on manipulative skills to an emphasis on developing concepts, relationships, structures, and problem-solving skills." This could drastically modify the content of the mathematics curriculum, in all grades, especially in grades 7-12.

These reports were all used as resources by the Wisconsin Mathematics Curriculum Task Force. The recommendations of this guide are compatible with the recommendations of these national reports. Curriculum developers in local school districts are urged to consider the source of the recommendations and reject them only on the basis of a compelling rationale.

Assumptions

The mathematics curriculum must continuously change. Since text materials are generally replaced every five to seven years, it is appropriate to think of curriculum development as having a cycle which corresponds to the textbook adoption cycle (see "The Curriculum Development Process," section 10). With this in mind, the Wisconsin Mathematics Curriculum Task Force has developed these recommendations with the expectation that they will serve reasonably well for the period from 1985 to 1990. During that time it will be necessary to prepare new recommendations for the period from 1990 to 1995.

Students entering first grade in 1985 will graduate from high school in 1997, and will retire from the work force in about 2044.

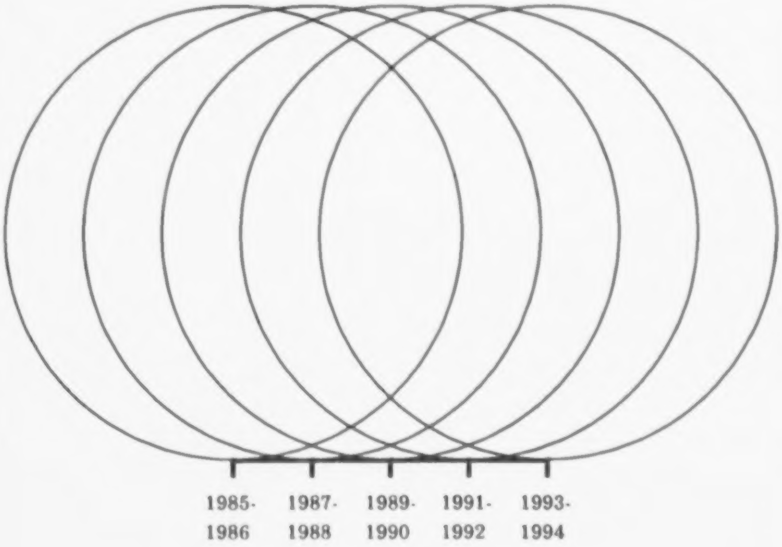
Students entering first grade in 1985 will graduate from high school in 1997, and will retire from the work force in about 2044. A contemporary curriculum would prepare students to be productive adults throughout that period. But there are two reasons a truly contemporary curriculum is not possible. The first is the inability of educators to know what mathematics will be needed by productive adults in 2044. It is hard enough to know what mathematics will be needed by productive adults in 1990. The second is a practical problem of implementation. Teachers, administrators, students, and parents become accustomed to the curriculum as it is and are uncomfortable when drastic changes are made. Certainly the curriculum of 2044 will be drastically different than the curriculum of today.

To address these two problems, the task force has assumed that curriculum should be modified through a large number of small changes rather than a small number of large changes.

By making incremental changes over a period of years the curriculum can be substantially modified without causing a major disruption among teachers, students, and parents. In figure 1, each circular region represents the mathematics topics covered in a particular year. A planned sequence of small changes produces a very different curriculum after ten years. Districts which find the recommendations of this guide very similar to their present curriculum may want to move rapidly to implement these recommendations. Districts which find these recommendations substantially different from their present curriculum will probably want to think of them as goals for 1990 (or later) and make a series of smaller systematic changes each year as a strategy for implementation.

Figure 1

An Evolving Curriculum



While it is difficult to foresee the future, the increasing role of calculators and computers in mathematics cannot be denied. Visionary mathematics educators have reported on the expected impact of computing technology on mathematics curriculum and instruction (Corbitt and Fey, 1984; Fey, 1984; Hansen, 1984). Execution of standard computational procedures and the manipulations of symbols in algebra become trivial applications of technology. Their role in the curriculum should be minimized. Teachers should "shift the focus of instruction from an emphasis on manipulative skills to an emphasis on developing concepts, relationships, structures, and problem-solving skills" (Corbitt and Fey, 1984). The task force supports this view and has chosen to move cautiously in this direction.

Teachers should "shift the focus of instruction from an emphasis on manipulative skills to an emphasis on developing concepts, relationships, structures, and problem-solving skills."

Mathematics is most effectively learned through a spiral approach in which a concept is introduced, subsequently developed through one or more lessons (possibly over several years), and finally mastered. Thereafter, periodic review is necessary to maintain mastery. It is not desirable to indicate this spiral development by listing topics at every grade level, for it results in the needless repetition of objectives and adds unnecessarily to the bulk of the guide. Generally, objectives are listed only once and not repeated as review objectives. Review and maintenance of concepts and skills are required in later grades for all objectives, so all objectives should be considered for subsequent grades. The intention of the task force is that all objectives identified for grades K-8 will be taught to all students, that each student will master as much as possible, and that periodic review will enable all students to retain the mastery they achieve throughout the program.

Purposes of the Guide

In organizing and preparing this guide the task force has assumed that the audience consists of members of district mathematics curriculum committees. This guide will help the district mathematics curriculum committee develop a district curriculum guide, which is the guide teachers in the district should use in their planning and decision making.

Consistent with this assumption, instructional objectives have been organized so as to reveal the continuity of development throughout a strand extending from kindergarten through grade 12. It is not intended that a fourth grade teacher, for example, should be able to open the guide to a section describing the fourth grade curriculum. The district mathematics curriculum committee may choose the latter form of presentation for the convenience of the district's teachers, or it may choose, as this guide's task force has, to emphasize the continuity of topics by using a strands approach.

The task force has avoided making specific recommendations for specific grade levels since textbooks vary somewhat in the grade-level placement of topics. Generally, it is most convenient to teach a topic at the time it occurs in the text. Thus, objectives have been organized by clusters of grade levels to minimize any confusion on this point.

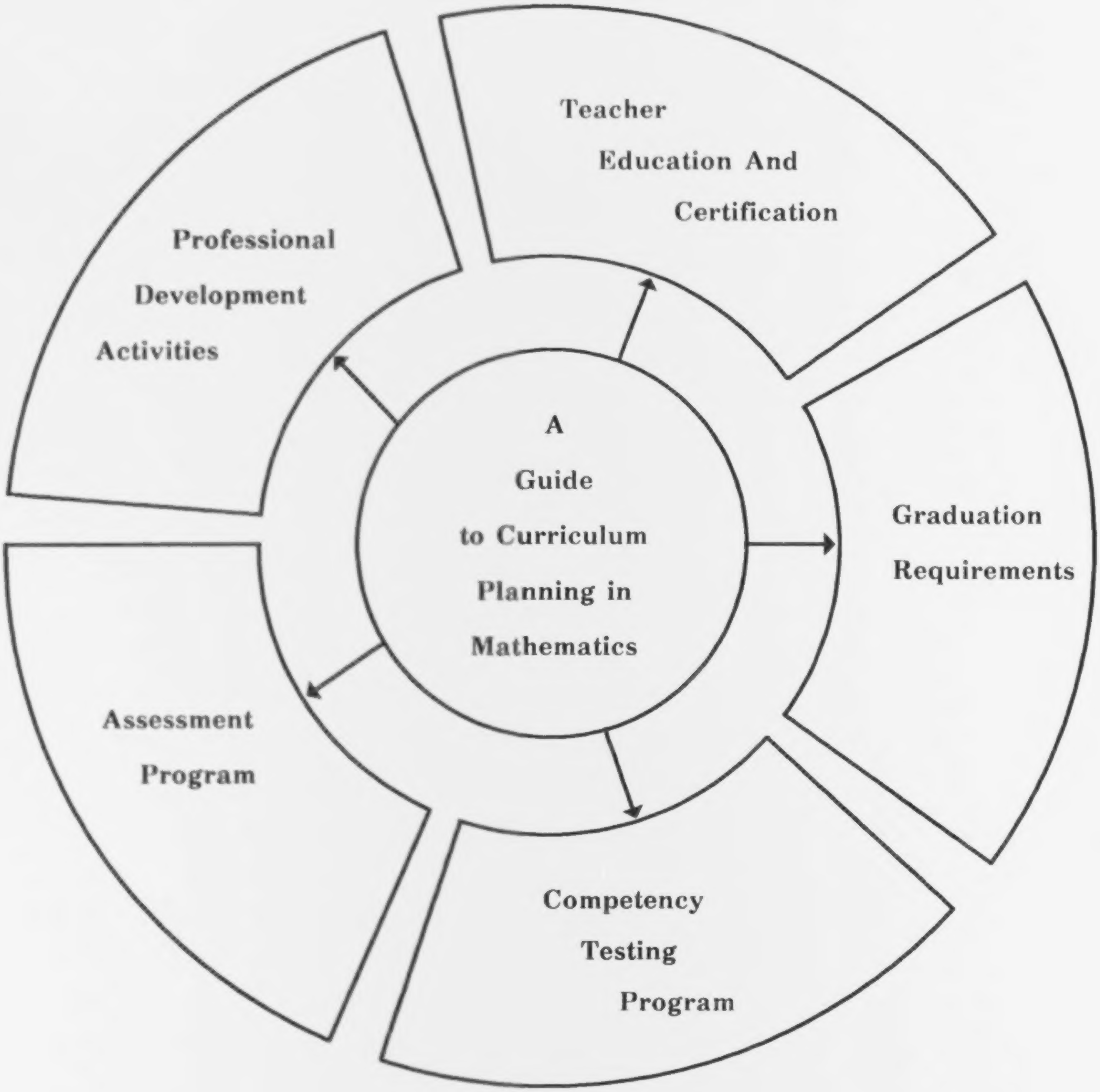
In preparing its recommendations the task force has relied extensively on major reports of commissions, study groups, and professional organizations. Many of these reports, or excerpts from them, are included in the guide's appendixes. These reports generally are quite compatible. There are, however, some issues on which some reports conflict. The task force has made its recommendations with full awareness of these conflicting recommendations, and assumes that local curriculum committees will do the same.

The guide . . . serves as a model and resource to committees working at the local level to develop their own curriculum guide.

The guide has been developed to serve four purposes. First and foremost it serves as a model and resource to committees working at the local level to develop their own curriculum guide. As such it helps to avoid duplication of effort in each of Wisconsin's 432 school districts. It is hoped that the model provided will stimulate local curriculum development and subsequently result in improved curriculum and instruction. Second, it suggests criteria for content selection, and strategies for curriculum articulation, implementation, and evaluation once the curriculum has been identified by the committee. A third purpose is to provide a description of the K-12 mathematics program for pre-service teachers in teacher education programs. And, finally, this guide is intended to serve as a basis for all mathematics-related activity at the Department of Public Instruction, including the Mathematics Assessment Program, the Mathematics Competency Testing Program, and the high school graduation requirement in mathematics.

Figure 2 indicates the central role played by the curriculum guide in planning and implementing other mathematics-related projects at the Department of Public Instruction.

Department of Public Instruction Mathematics-related Projects


Teacher
Education And
Certification
Professional
Development
Activities
A
Guide
to Curriculum
Planning in
Mathematics
Graduation
Requirements
Assessment
Program
Competency
Testing
Program

This central role of the curriculum guide assures continuity and consistency among all projects. We suggest that a similar strategy be employed in local districts.

The local district guide, once developed, serves to communicate the curriculum to teachers, administrators, school boards, and the general public.

Structure of the Guide

The guide is organized into 12 sections. The Introduction provides a background for using the guide for curriculum development and states the philosophy.

Sections 2 through 8 are the heart of the guide and cover the goals and objectives of an exemplary mathematics program. The instructional objectives have been organized into seven strands, none of which is entirely exclusive of the others. The strands are: number and numeration, measurement, geometry, statistics, algebra, discrete mathematics, and problem solving. Objectives in each strand have been identified for groupings of grade levels. The groupings are: K-3, 4-6, 7-8, and 9-12. Objectives introduced at earlier grade levels have not been restated for later grade levels, even though periodic review is necessary. The curriculum for any grade level is, in reality, the combination of the objectives stated for that grade level and all those from previous grade levels.

Rather than focus attention on the college preparatory objectives at the high school level, the decision was made to define a program for all students in sections 2 through 8 and defer discussion of the college preparatory program to section 9. Some attention to organizing programs for diverse student interests and needs also is given in section 9.

Section 10 provides a description of the curriculum development process. Section 11 contains a brief description of some important issues in mathematics education which affect curriculum selection and instruction. Section 12 lists resources. The appendixes include original source materials for reference and documentation.

Highlights

The recommendations of this guide are supported by the reports mentioned in the Introduction, many of which are included in the appendixes. Some of the most important recommendations are highlighted here.

Numeration Concepts Should Receive More Attention

Difficulties with whole-number and decimal computation are frequently a result of a premature emphasis on algorithmic computation. The algorithms rely on place-value concepts. If place-value concepts are inadequately understood, procedures for adding, subtracting, multiplying, and dividing don't make sense and are more difficult to recall. The result is frequently the invention of faulty, or "buggy algorithms" by the student. Buggy algorithms are especially troublesome with fractions. Here students are inclined to treat a fraction as an arrangement of whole numbers rather than as a new kind of number. Students focus on manipulating the parts (numerators and denominators) without learning about rational numbers which have properties independent of the properties of their whole-number components. In addition to more extensive study of place value, exponential notation should be introduced and developed in the middle school grades. Exponential notation provides a useful way to deal with very large and very small numbers, and allows calculators to treat numbers larger than a standard display can hold.

See supporting documentation in Appendixes D, E, G.

Less Attention Should Be Given to Pencil and Paper Computation with Whole Numbers, Decimals, and Fractions

School mathematics programs are very successful in teaching whole number computation. Some time now devoted to developing that skill should be allocated to other important activities. There is still a need to develop knowledge of basic facts and algorithms for small numbers. Teachers should also demonstrate the versatility of the algorithms by applying them to larger numbers, but only a minimum amount of classroom time should be devoted to this and student mastery should not be expected. Addition and subtraction should be mastered for numbers of three or fewer digits or special cases involving more than three digits. Multiplication should be mastered only for one- or two-digit numbers. Division should be mastered for one-digit divisors or one-digit multiples of ten or one hundred.

In the early stages, computation with fractions should be restricted to commonly encountered fractions with denominators of 2, 3, 4, 5, and 10. When adding and subtracting fractions there is little point in using relatively prime denominators except those just listed. Later, as students deal with ratios in which a greater variety of denominators might occur, and as students prepare for algebra, where rational expressions such as x/y are treated, fractions with relatively prime denominators assume more importance.

See supporting documentation in Appendixes B, C, D, E, F, G.

More Emphasis Should Be Given to Estimation, Mental Arithmetic, and the Use of Calculators

Children should be taught to compute efficiently. They should understand that different methods are appropriate in different situations. Most people would find mental arithmetic most efficient for 60 × 40 and calculators most efficient for 6485 × 4762. To provide children with appropriate options, programs should teach how to estimate, how to do mental arithmetic, and how to use calculators. Mental arithmetic is not the same as executing the pencil and paper algorithm mentally. Different strategies are used. For example, 5% of $3.80 (the tax on a $3.80 lunch) is not $.05 \times 3.8$, but $.1 \times 3.8 \div 2$. Because pencil and paper algorithms are seldom the most efficient for large numbers, less emphasis on computation with large numbers was recommended above.

See supporting documentation in Appendixes B, C, D, E, F, G.

More Emphasis Should Be Given to Measurement, Geometry, Statistics, and Problem Solving at all Grade Levels

Many important aspects of mathematics do not have computational prerequisites and should be introduced when mathematics instruction begins. To the extent possible, all mathematics should be developed in a problem-solving mode. Measurement is one of the most frequently encountered applications of mathematics. It provides a versatile application for arithmetic and is an important component of geometry. Children must learn more geometry than recognizing shapes. They should also learn properties of figures (including, but not limited to measurement properties) and relations among figures. Geometric concepts in both two and three dimensions should be taught.

Instruction in statistics is currently unavailable to virtually all students at all grade levels. The need for statistical knowledge by all persons has increased dramatically during the last few years, and new approaches to the teaching of statistics have been developed. Topics from exploratory data analysis should be taught to students of all ages and ability levels.

Problem solving should include, but not be limited to, routine story problems requiring one or more applications of addition, subtraction, multiplication, or division. Children should also learn problem-solving strategies which can be applied to nonroutine problems, which are different from those they have encountered previously. Algorithmic thinking is becoming more and more an integral part of mathematical problem solving. Procedures, and especially iterative (repetitive) procedures associated with computer methods of solving problems should be used regularly.

See supporting documentation in Appendixes B, C, D, E, F, G.

All Students Should Take at Least Three Years of Mathematics in Grades 9-12, and at Least 50 Minutes Per Day Should Be Allocated to Mathematics Instruction

A new Wisconsin law requires all students graduating after September 1, 1988, to complete at least two years of mathematics in grades 9-12, including some arithmetic, some algebra, some geometry, and some statistics. However, the National Council of Teachers of Mathematics has recommended that at least three years be required, and the National Science Board Commission on Precollege Education in Mathematics, Science and Technology has recommended that all students take three years, including one year of algebra.

The National Council of Teachers of Mathematics has recommended a minimum of one hour per day of mathematics instruction in primary grades and seven hours per week in the upper grades. The National Science Board Commission on Precollege Education in Mathematics, Science and Technology recommends a minimum of 60 minutes per day. The Wisconsin Department of Public Instruction has determined that at least 50 minutes per day can be allocated to mathematics instruction without restricting the amount of time available for instruction in other curriculum areas (see table 16, p. 129).

In order to meet the recommendations of this guide and the Wisconsin graduation requirements, districts are encouraged to require not less than one year of algebra, one semester of geometry, and one semester of statistics. By one year of algebra we do not mean the traditional first-year algebra course. This course is still appropriate for many students planning to enter college, but is not appropriate as a graduation requirement for all students. It has been common in recent years to extend the time allocated to the first-year algebra course to two years so that many less well-prepared students can complete it at a more leisurely pace. This is not appropriate either since many topics in that course are inappropriate for students not planning to enroll in the second year algebra course. For those students, the recommended program includes one year of selected topics from the first-year algebra course.

By recommending one semester of geometry as a requirement for graduation we do not mean that college-bound students should take one semester instead of one year. Those students should continue to take a full year of geometry or else risk being denied admission to the college of their choice. Furthermore, we do not mean taking the first semester of the standard geometry course. Rather we mean a new course which focuses on many of the most important and commonly applied results approached from an intuitive point of view with little or no time devoted to formal proof.

See supporting documentation in Appendixes C, E, F, G.

Discrete Courses in Algebra and Geometry Should Be Replaced by Integrated Courses

Course content in grades K-8 is generally integrated; that is, some arithmetic, some geometry, some algebra, and some statistics are included at each grade level. The content is ordered according to difficulty with easier topics preceding harder topics, and topics which complement each other can be conveniently related. Based on pedagogical considerations, this format is preferable to the current one for grades nine and ten which commonly includes one year of algebra followed by one year of geometry. This recommendation does not necessarily affect the content of the curriculum. It is an organizational feature which is intended to facilitate instruction.

See supporting documentation in Appendixes E, F.

Philosophy

This statement is intended to express the philosophy of mathematics instruction held by the members of the task force that developed the mathematics curriculum guide. It also provides a model philosophy statement which district mathematics committees can adopt, adapt, or otherwise respond to in the preparation of their own philosophy statements.

The over-arching goal of education is to develop informed, thinking citizens capable of making decisions on personal, community, national, and world issues. The mathematics program develops skills and knowledge of concepts necessary for decision making in personal living and also provides the background for a variety of educational and vocational options, including preparation for further study of mathematics. The program also provides an appreciation of the role of mathematics in history and culture and an understanding of the nature of abstract structures as generalizations of a variety of more specific structures.

The most important goal of mathematics instruction is the development of students' ability to solve problems. Problems may be of two kinds. Routine problems are of a type frequently encountered and for which standard procedures are taught and practiced. Nonroutine problems require more than immediate recall or the use of previously established patterns of action. Because it is impossible to anticipate all the future needs of children, the school mathematics program should provide a balanced emphasis on recall of facts and definitions, use of algorithms, and routine and nonroutine problem-solving strategies. It should not focus exclusively on the acquisition of specific skills and procedures.

Preparing students for advanced work in mathematics continues to be an important goal of school mathematics programs. However, most high school students do not eventually study calculus. The program should continue to accommodate the needs of students having the talent and inclination to prepare for calculus and other college or university level mathematics courses, but preparation for calculus is not appropriate as the framework for the entire school mathematics program. An equally important goal is the mathematics education of average and below-average ability students, many of whom will not study mathematics after high school. All students should be able to solve routine problems and simple nonroutine problems in measurement, statistics, geometry, and algebra. Every student should also have a working knowledge of applications of arithmetic.

The mathematics program should recognize the widespread use of calculators and computers outside the classroom and should teach mathematics in such a way as to take advantage of their power and convenience. Instructional time devoted to skills that replicate functions of calculators and computers should be minimized.

Evaluation of student performance by teacher-made tests, competency tests, or norm-referenced tests should reflect the priorities identified for mathematics teaching and learning. All tests should be selected or designed to be compatible with curriculum priorities. While it is easier to assess performance on lower order computational skills than on higher order analysis and problem-solving skills, every effort should be made to maintain the same balance in tests that exists in the curriculum. Furthermore, evaluation should be based on classroom observations and discussions as well as written tests.

Student success in mathematics depends not only on good curriculum but also on good teaching. Teachers should possess a thorough knowledge of the mathematics being taught, understand why that mathematics is being taught, and be able to select from a variety of instructional strategies for each concept or topic. Since student achievement in mathematics depends to a large degree on the amount of instruction in mathematics, students at all grade levels should have access to fifty minutes or more of mathematics instruction per day. Teachers should maximize the time each student is successfully engaged in learning mathematics.

In a quality mathematics program comparisons of enrollments and achievements using sex, ethnic origin, or socio-economic status as a variable will show no differences.

The importance of mathematics is independent of a student's sex, ethnic origin, and socio-economic status. Furthermore, the ability to learn mathematics is not genetically related to these factors. In a quality mathematics program, comparisons of enrollments and achievements using sex, ethnic origin, or socio-economic status as a variable will show no differences. All programs should aggressively seek to eliminate any such differences that may exist.

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2

Number and Numeration Overview

Quantitative thought and understanding are becoming increasingly important to function effectively in modern society and in the workplace. As a consequence, arithmetic continues to occupy a central place in the elementary mathematics curriculum. But the importance of various aspects of arithmetic has changed and will continue to change as the computational power of electronic devices becomes more widely accepted and widely used in schools.

This is not to suggest that computational proficiency is no longer a legitimate goal of mathematics instruction. But one must recognize that there are several ways to compute—mental computation, paper-and-pencil algorithms, estimation, and use of calculators and/or computers. Today's students must become adept at all these methods, and more importantly, be able to decide which method is most appropriate for a particular problem or situation and what degree of precision and accuracy is required.

Thus, the focus of arithmetic instruction should be on the effective use and understanding of numbers in applications as well as other mathematical contexts and not simply on attaining high levels of skills and efficiency with paper-and-pencil computational algorithms. Excessive drill and practice with formal paper-and-pencil computations, especially with larger numbers, must be replaced with activities and procedures that demonstrate the utility of mathematics in solving a wide variety of problems.

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Mathematical Structure

Research is beginning to document the critical importance of structure and understanding in learning arithmetic. Number systems and the arithmetic operations on numbers should not be taught as isolated concepts or skills. Children need activities to develop better number sense. Instruction should emphasize the structure of the number systems and relationships among numbers in order to provide a framework for students to apply the concepts and skills they learn.

Children's Informal Knowledge

There is a growing body of evidence that students develop a great deal of informal knowledge about numbers independent of instruction, particularly at the primary grade level. Much of this comes through a variety of counting experiences. Teachers should assess students' knowledge in order to build upon it when it is sound and specifically address misconceptions that may have developed. Teachers should be sensitive to the processes students use to get answers, as well as the answers themselves. Students frequently invent algorithms that are different from those taught. In some cases, the procedures are quite efficient and provide a basis for furthering understanding. In other cases, students consistently use incorrect algorithms. In the latter case, students need to be told more than that their answer is wrong. Measures must be taken to address the specific misconceptions leading to the inappropriate algorithm.

Computation

Facility with one-digit number facts is as important as ever, but drill on the arithmetic operations with larger than three-digit numbers should be de-emphasized. Such computations can and should be done by calculators and computers. The selective use of calculators and computers should be encouraged, both to help develop concepts and to do many of the tedious computations that previously had to be done using paper and pencil.

. . . drill on the arithmetic operations with larger than three-digit numbers should be de-emphasized.

Students need to develop a feel for the magnitude of numbers and understand the relationship of numbers to physical situations that they represent. A thorough understanding of place value and numeration is necessary to provide meaning to algorithms. Students also need to develop an understanding of the algorithms themselves. Appropriate manipulatives should be used to give meaning to algorithms. When using manipulatives the correspondence between the physical manipulations and the written algorithm should be stressed. Students should also learn to estimate an answer and should regularly use estimation to check the reasonableness of their answers.

Fractions and Decimals

Working with numbers and number operations should not be carried out in the abstract. Operating within a real-world, problem-solving context gives meaning to numbers and to the operations on them. This is particularly important for fractions and decimals. The need for non-whole numbers should arise within a measurement context so that children can come to realize that fractions and decimals are, indeed, numbers that behave in much the same way as whole numbers. The understanding of fractions and decimals as numbers, comparison of fractions and decimals, and conversions between fractions and decimals should receive more emphasis, while drill on addition, subtraction, multiplication, and division of fractions with large denominators and multi-place decimals should be given less emphasis.

Mental Arithmetic

Mental arithmetic should be emphasized at all levels, first aimed at exact answers and later at approximate ones. Mental arithmetic involves finding natural and easy, not formal algorithmic, ways of looking at number relationships. There should be heavy and continuing emphasis on estimation, not only in formal round-off procedures, but in developing a feel for numbers. Students need experience in estimating actual quantities (measures) as well as in estimating numerical quantities (computations) that appear in complicated form.

The Role of Calculators

Calculators should be used to develop concepts as well as eliminate tedious computation. With calculators students can solve problems before they have mastered computation skills. This helps to provide meaning for the operations. Calculators also make it possible to use realistic numbers in problems. They are also appropriate tools to use in developing concepts like factorization or square root.

Problem Solving

Problem solving integrated throughout the curriculum and instruction should emphasize the application of arithmetic skills to solve problems. Throughout this guide, the emphasis is on the inter-relatedness of the different strands of mathematics. Number and numeration are no different. It is most appropriate to draw realistic applications and problems from the domains of measurement and statistics. Problem situations provide meaning for arithmetic operations, and instruction can portray arithmetic as a system for representing and solving a variety of types of problems. Applications should always stress good problem-solving techniques, not just guessing which arithmetic operation should be used. The importance of different real-world applications of mathematics will change with time. Applications should be designed to develop general skills for applying mathematics to new situations, not to drill on particular applications that are likely to become outmoded.

Problem solving integrated throughout the curriculum and instruction should emphasize the application of arithmetic skills to solve problems.

Enrichment

Enrichment should involve the use of arithmetic in applied situations, not additional practice in computation. More capable students should be encouraged to develop their understanding of basic concepts and their problem-solving abilities, not their computational proficiency. Decisions about grouping for enrichment should not be made strictly on the basis of computational skill.

Table 1 ■

Number and Numeration Objectives

Number Concepts

K-3	**N1.** Identify the number of objects in a set by counting or estimating. Decide whether estimation or counting is appropriate. **N2.** Compare the number of objects in different sets. Order whole numbers.	**N3.** Identify fractions that represent parts of a whole or parts of a set for fractions with small denominators. **N4.** Count by multiples of 2, 5, 10, 100.
4-6	**N5.** Estimate the number of objects in large sets. Decide what level of accuracy is necessary for particular situations. **N6.** Understand different interpretations of fractions, such as parts of a whole, quotient, operator, and number line.	**N7.** Estimate fractions that approximate fractional parts of a whole. **N8.** Order common fractions. **N9.** Order decimal fractions.
7-8	**N10.** Use decimals to estimate quantities. **N11.** Relate integers to the number line and to appropriate physical situations.	**N12.** Order integers and find absolute value. **N13.** Understand the meaning of percent as a special case of a ratio involving hundredths.
9-12	**N14.** Understand distinctions between rational and irrational numbers. Give examples of rational and irrational numbers.	

Numeration, Place Value, and Equivalent Representations

K-3	**N15.** Read and write numerals and number names written in words for whole numbers less than 1,000. **N16.** Understand basic place value concepts for whole numbers. Identify the number represented by a given digit; write numbers in expanded notation (e.g., 500 + 20 + 3).	**N17.** Decide what place value is appropriate for particular situations and round numbers to that place value. **N18.** Read and write numerals for simple fractions.
4-6	**N19.** Read and and write numerals and number names written in words for whole numbers up to one billion. **N20.** Read and write numerals for fractions and decimals. **N21.** Understand the concept of equivalence of fractions and be able to illustrate it with appropriate materials or problem situations. **N22.** Generate equivalent fractions, change from improper to mixed fractions, and back, and reduce fractions. Solutions of problems should generally not require denominators greater than 24 at any step of the problem.	**N23.** Relate decimal fractions to common fractions and vice versa. This includes procedures for exact translation and techniques for estimating approximations. Recall decimal equivalents of frequently used fractions (1/2, 1/3, 1/4, 1/5, 3/4). **N24.** Understand basic place value concepts for decimals. Identify the number represented by a given digit; write numbers in expanded notation (e.g., 3 + 5/10 + 7/100). **N25.** Round decimals to an appropriate number of digits.

Table 1 *(continued)*

Number and Numeration Objectives

Numeration, Place Value, and Equivalent Representations (continued)	
7-8	**N26.** Translate fractions and decimals to percents and vice versa. Recall fraction equivalents of common percents 50%, 25%, 75%, 33-1/3%, 10%, 20%, 100%. Some exposure should be given to percents involving decimals such as 8.5%, but the emphasis should be on understanding basic percent concepts, not computation. **N27.** Read and write integers. **N28.** Read and write numbers in scientific notation. Identify appropriate contexts for scientific notation. Identify appropriate number of significant digits for scientific notation. **N29.** Understand the meaning of the term "significant digits."
9-12	**N30.** Write rational numbers as fractions or repeating decimals.
Number Operations	
K-3	**N31.** Use objects, pictures, and problem situations to model and interpret different definitions of addition, subtraction, multiplication, and division of whole numbers. **N32.** Accurately and with minimal hesitation provide response for basic addition and subtraction number fact combinations. **N33.** Understand the algorithms to find the sum and difference of whole numbers and choose the appropriate method of calculation (estimation, mental arithmetic, paper and pencil, or calculator). Students should quickly find sums and differences for numbers with one significant digit. They should identify the appropriate contexts and number of significant digits for estimates and make them quickly both mentally and using paper and pencil. Drill on paper and pencil calculations should emphasize no more than three-digit numbers except in special cases such as calculations involving zeros. Addition should generally involve no more than four addends. **N34.** Write number sentences to represent problems involving different addition, subtraction, and multiplication situations, and solve the sentences. (See objectives A2 and A4.)* **Cross-references are made among the mathematics curriculum strands using the identifying letter and number of each objective.*
4-6	**N35.** Accurately and with minimal hesitation provide response for basic multiplication and division number fact combinations. **N36.** Understand the algorithms to find products and quotients of whole numbers and choose the appropriate method of calculation. Understanding includes knowing how to deal with the remainder in different division problem contexts. Students should be proficient in mentally finding products and quotients for numbers with one significant digit. They should identify the appropriate contexts and number of significant digits for estimates and make them quickly both mentally and using paper and pencil. Students should be given some experience with three-digit factors so they learn general algorithms for multiplying. However, drill on paper and pencil calculations for multiplication should emphasize products of no more than two two-digit numbers except in special cases such as calculations involving zeros. Division should generally emphasize calculations with one-digit divisors or a divisor of one significant digit with quotients of three digits or less except for special cases. **N37.** Using appropriate physical or pictorial materials, illustrate the meaning of addition, subtraction, and multiplication of fractions and decimals. Identify problem contexts that represent different interpretations of addition, subtraction, and multiplication.

Number Operations *(continued)*

4-6 *(cont.)*	**N38.** Understand the algorithms to find sums, differences, and products of fractions and decimals and choose the appropriate method of calculation such as estimation, paper and pencil calculation, changing to a decimal, or using a calculator. Students should develop a feel for the magnitude of calculations and use estimation to check calculations. Calculations with fractions generally should not require denominators greater than 24 at any step of the problem. Decimal calculations should emphasize understanding the algorithms and being able to use them for all situations. Calculations with numbers having many digits should be de-emphasized. **N39.** Write number sentences to represent problems involving single and multiple operations involving whole numbers, fractions, and decimals and solve the sentences. (See objective A4.)
7-8	**N40.** Understand basic properties of number operations (commutative, associative, distributive, identity, etc.) and use them to simplify problems and to perform mental calculations. **N41.** Understand the algorithms to find quotients of fractions and decimals and choose the appropriate method of calculation. (See objectives N36 and N38 above.) **N42.** Using appropriate materials and/or problem situations illustrate the meaning of addition, subtraction, multiplication, and division of integers. **N43.** Understand the algorithms to find sums, differences, products, and quotients of integers. **N44.** Evaluate expressions involving knowledge of grouping symbols and order of operation. **N45.** Use ratio and proportion to solve problems. **N46.** Use rules of divisibility for 2, 3, 4, 5, 9, 10, and their multiples. **N47.** Identify odd, even, prime and composite numbers, and perfect squares. Know all primes less than 30 at a recall level. **N48.** Identify multiples, factors, prime factors, least common multiple, and greatest common factor and understand their importance with respect to other arithmetic contexts. Understanding, not computation, should be emphasized. Calculators can be used to provide experiences with larger whole numbers to eliminate tedious computation. **N49.** Solve basic percent problems (all three cases included). Percents are primarily an application of mathematics and applications should be emphasized, not calculations. Students should choose the appropriate method of calculation (paper and pencil calculation, estimating, or calculator) and the appropriate level of accuracy. The emphasis should be on understanding, not proficiency with paper-and-pencil calculation. Calculators should be used for problems requiring tedious calculations.
9-12	**N50.** Estimate square roots to two significant digits. The emphasis should be on understanding recursion, not calculation. Calculators should be used for tedious computation. **N51.** Simplify expressions involving roots and exponents.

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3

Measurement Overview

Measurement is the process of assigning a number to some entity—an object, a set of objects, or an event—on the basis of an identifiable attribute and some unit that has the same attribute. Measuring includes acts such as counting (a set and the attribute of numerosity), determining a perimeter (a geometric object and the attribute of the length of its sides), and assigning a probability (an event and the attribute of likelihood of occurrence). This pairing of a number with an entity is an example of a function, one of the most powerful ideas in mathematics.

Measurement is an excellent example of the interrelationships of the various strands of mathematics and of the interrelationships of mathematics and other disciplines. Measures have application in arithmetic, algebra, geometry, probability, and statistics. Measures and measurement are also central to the natural sciences, social sciences, fine arts, and industrial arts.

Problem Solving

The most important goal of mathematics instruction is the development of students' ability to solve problems. Students should learn to think of their world as describable in logical, quantitative, and geometric terms. It is difficult to think of applied problems in which the relevant numbers—price, time, age, cost, mass, distance, etc.—are not measures or numbers derived from measures.

It is difficult to think of applied problems in which the relevant numbers—price, time, age, cost, mass, distance, etc.—are not measures or numbers derived from measures.

Measuring should be learned not for its own sake but rather as a powerful tool for solving realistic and interesting problems. This strand should include more than just geometric attributes and formulas. Time, temperature, mass, liquid capacity, and monetary value are examples of other attributes that should be included. Pupils must be given extensive and varied opportunities to use measurement in a natural and active way, to develop the necessary skills and understandings through a well-balanced curriculum that progresses through many levels of sophistication and complexity.

Measurement Systems

Measurement should not be thought of or taught as a collection of narrow, unrelated skills such as reading a ruler, a gauge, or protractor or using several memorized formulas for determining measures of geometric objects. Rather, it is more appropriately characterized as a system of concepts and ideas, skills and applications that should be taught and learned as a system, regardless of whether the units used belong to the metric system or the U.S. Customary System. Children should have equal exposure to the standard units of both the metric and U.S Customary systems. In order to gain familiarity with units and their relative sizes, comparisons between the two systems are expected. However, there should be no attempt to have students carry out formal computational conversions between the two systems.

Active Measuring

The process of measuring is best learned through active experiences over all the school years.

The process of measuring is best learned through active experiences over all the school years. During early measurement instruction children should be exposed to physical activities that allow them to become familiar with the attributes of objects to be measured. Next they learn to discriminate among those attributes and to directly compare objects on the basis of quantitative properties in relational terms such as "is equal to," "is less than," or "is greater than." Once direct comparison becomes impossible or impractical, children learn to make representations of the objects being compared. A good way to represent objects is by measuring, first with arbitrary units (e.g., the width of a hand to measure distance). Experience with arbitrary units should lead students to discover the merits of selecting more widely accepted units of measure and establish the need for standard units, either metric or U.S. Customary.

Units of Measure

In learning measurement with standard units, students should begin with tools that are simple to use and understand and gradually move toward those that are more complex. Several key ideas are embedded here. First, the tools are based upon "automatic" iteration of a unit and the fact that congruent objects (here units) have the same measure. Second, as larger numbers of units are needed, the utility of grouping is emphasized. This has direct application to the ideas of place value. Third, the important idea of appropriateness or choice of unit comes in. That is, large units should be used to measure large objects and small units for small ones. And finally, the need to sub-divide units and use fractional parts of whole units will arise. This use of number within a measurement context demonstrates the power of the overall measurement idea in making certain numerical concepts (in this case, fractions) more understandable.

Approximation

The approximate nature of measurement also should be taught. Pupils should learn that in making and recording measurements, they are working with approximations. The physical act of measuring (e.g., using a ruler or tape measure to find the length of a segment) produces, at best, an approximate measure due to the limitations of the ability to read the instrument and to the degree of precision of the instrument itself. Within this context, the process of rounding off becomes both useful and meaningful, not only to numbers derived from measuring but also to numbers presented in arithmetic exercises.

Estimation

A closely related idea is that of estimation, an ability that has great practical value in measurement. In the early grades, estimates of measurements should be encouraged through comparisons with already accepted measures, whether arbitrary or standard. In the later grades, as students become more familiar with standard metric or U.S. Customary units, they should learn to make reasonable estimates through direct observation or through estimating and rounding procedures with whole number computation.

Direct and Indirect Measurement

A clear distinction should be made between direct measurements that arise from use of a particular measuring device, such as a stop-watch, a scale, a thermometer, or measuring-cup and indirect measurements that are derived by use of other methods. One of these methods, estimation, has already been mentioned. Others are use of formulas, similar triangles and trigonometry, the Pythagorean theorem, or arithmetic processes as in derived measures like speed, acceleration, or density. These methods of determining indirect measurements demonstrate once again the interrelatedness of the measurement strand with other mathematics strands such as algebra, trigonometry, geometry, and number and numeration.

Formulas

Special mention should be made of the use of formulas. As useful as formulas may be in determining indirect measurements, they lose some of their utility if there are too many to be memorized and the student becomes confused or tends to use an incorrect formula. A useful procedure is to memorize only the most essential and most frequently used of formulas or those that are very difficult to derive, and then either reconstruct formulas by using general principles, look up the formulas in some reference book, or simply determine a measurement by thinking of the attribute that is to be measured. A good example is perimeter of polygons. If a student simply remembers that perimeter means to find the distance around, no formulas are needed, just the ability to add correctly. Basic formulas that probably should be committed to memory are circumference and area of circles, area of triangles, rectangles, and parallelograms, and volume of a rectangular solid. In any case no formula should be introduced until students fully understand the measure that the formula enables them to calculate.

. . . no formula should be introduced until students fully understand the measure which the formula enables them to calculate.

Calculators

The selective use by students of electronic computing devices, particularly the hand-held calculator, should be encouraged both to help develop concepts and to do many of the tedious computations that previously had to be done with paper and pencil. The use of the calculator can permit students to deal with realistic problems and with data that they have generated themselves through direct, active measurement. Interesting activities need not be postponed because of lack of requisite computational skills, say with decimals, nor do students need to be bound to textbook exercises that may lack reality or interest.

Table 2

Measurement Objectives

Measurement Concepts	
K-3	**M1.** Organize techniques related to measuring using selected arbitrary units of measure. **M2.** Become familiar with attributes of length, weight, area, liquid capacity, time, temperature, and monetary value. (See objectives G19 and G20.) **M3.** Explore the nature of the measurement process. This implies exposure to the approximate nature of measurement, being able to choose an appropriate unit of measure, and selecting proper measuring instruments.
4-6	**M4.** Become familiar with attributes of perimeter, volume, angle measure, duration, and likelihood of occurrence. (See objective G20.) **M5.** Understand the nature of the measurement process. The various notions mentioned in the K-3 section should be continued and reinforced especially with respect to some of the newer attributes. In addition, the student should be introduced to other aspects such as precision of measurement, the role of estimation, and the distinction between direct and indirect measurement. **M6.** Understand the conversion-of-units process as the reassignment of a new measure to the same entity with respect to the same attribute, based upon a different unit of measure. The inverse relationship between the size of the measure and the size of the unit should be explored.
7-8	**M7.** Understand the role and use of formulas and how they are developed. This includes an appreciation of using letters as variables and how the substitution process can be used to evaluate formulas. (See objective A7.) **M8.** Extend earlier notions of perimeter and area to include the specialized attributes of circumference and surface area. (See objective G23.) **M9.** Understand the indirect measurement process involved in such procedures as use of formulas, similar triangles, scale drawings, the Pythagorean Theorem, and simple right triangle trigonometry. **M10.** Understand the notion of precision of measurement and how it relates to the concept of the approximate nature of measurement. The role of significant digits should be examined within this context. **M11.** Understand the metric system. This includes the decimal relationship among the various units for a particular attribute, such as millimeter, centimeter, decimeter, meter, etc. as well as the interrelationship among the gram, milliliter and cubic centimeter.
9-12	**M12.** Become familiar with the attributes that arise in the study of various sciences, such as rate (of growth), speed, acceleration, density, light intensity, and sound. **M13.** Understand the number π (Pi), its derivation, and its relationship to the diameter, radius, and circumference of a circle. (See objective G23.)
Direct Measurement Skills	
K-3	**M14.** Acquire the skills associated with length measurement. This includes the exposure to the following units: inch, foot, yard, meter, and centimeter; a knowledge of the relative size of one unit to another; an ability to read and write the requisite vocabulary words associated with them; and a familiarity with the abbreviations and symbols. These skills also include the ability to use a standard measuring instrument calibrated in one or more of these units and to read a measurement to the nearest half-inch or centimeter. (See objective G19.) **M15.** Acquire the skills associated with weight (mass) measurement. This includes the exposure to the following units: ounce, pound, gram, and kilogram; a knowledge of the relative size of one unit to another; an ability to read and write the requisite vocabulary words associated with them; and a familiarity with the abbreviations and symbols. These skills also include the ability to use an appropriate standard measuring instrument (scale or balance beam). For a calibrated scale, the child should be able to read the scale to the nearest ounce, pound, or multiple of 10 grams.

Direct Measurement Skills *(continued)*

K-3 *(cont.)*	**M16.** Acquire the skills associated with liquid capacity measurement. This includes the exposure to the following units: cup, pint, quart, gallon, and liter; a knowledge of the relative size of one unit to another; an ability to read and write the requisite vocabulary words associated with them, and a familiarity with the abbreviations and symbols. These skills also include the ability to use standard measuring devices to determine the liquid capacity of a container. Measurements should be determined to the nearest quarter-cup, half-pint, or half-liter. **M17.** Acquire the skills associated with monetary values of coins and bills. This includes all appropriate vocabulary, the recognition of all coins and bills up to $5.00, the knowledge of their value (coins in cents and bills in dollars), and the following equivalences: one dime ↔ two nickels, one quarter ↔ five nickels, one half-dollar ↔ two quarters, one dollar ↔ four quarters, and one dollar ↔ two half-dollars. The child should also be able to make change for a purchase less than $1.00. **M18.** Acquire the skills associated with time measurement. This includes all vocabulary skills—reading, writing, speaking, use of abbreviations and symbols—for calendar aspects of time, such as days, weeks, months, and years. Simple equivalences such as one week ↔ seven days, one year ↔ 52 weeks, and one year ↔ 12 months should be known. Similar skills are expected for clock aspects of time such as hours, minutes, and fractional parts of hours. With either regular or digital clocks, the child should be able to tell time to the nearest quarter-hour.
4-6	**M19.** Continue acquisition of the skills associated with length measurement. (This is similar to objective M14.) New units are the millimeter, decimeter, kilometer, and mile. Precision of direct measurement with a device should be to the nearest millimeter or sixteenth of an inch. (See objective G20.) **M20.** Continue acquisition of the skills associated with weight (mass) measurement. (This is similar to objective M15.) New units are the milligram, the ton, and the metric ton. Precision of direct measurement with a balance beam should be to the nearest gram. **M21.** Continue acquisition of the skills associated with liquid capacity measurement. (This is similar to objective M16.) New units are the milliliter, kiloliter, and fluid ounce. Precision of direct measurement should be to the nearest fluid ounce or milliliter. **M22.** Continue acquisition of the skills associated with time measurement. (This is similar to objective M18.) Tell time to the nearest minute. Determine the duration of events by using a stop watch or by noting beginning and ending time and then calculating. **M23.** Acquire the skills associated with angle measurement. Use a protractor to measure angles to the nearest degree and to draw an angle of given size. (See objective G20.) **M24.** Acquire the skills associated with direct area measurement. This includes placement of whole and fractional parts of unit squares on regions and the superposition of grids on regions and then determining the number of units contained in the covering. (See objective G20.) **M25.** Acquire the skills associated with temperature measurement. Read to the nearest degree a thermometer marked in either Celsius or Fahrenheit.
7-8	**M26.** Refine and improve all the direct measuring skills introduced at earlier grade levels.
9-12	**M27.** Acquire the ability to use measuring devices related to scientific inquiry in subjects such as earth science, environmental science, biology, chemistry, physics, home economics, agriculture, and industrial arts.

Measurement Objectives

Indirect Measurement Skills		
K-3	**M28.** Acquire the skill of estimation. In early stages this involves a comparison with familiar objects within the range of a child's development. Later it involves the procedure of making an educated guess and then checking that guess against a measurement, as well as selecting an appropriate object or unit, either arbitrary or standard, as the basis for comparison and estimation.	**M29.** Solve simple one-step verbal arithmetic problems posed within a measurement context. **M30.** Determine perimeter of a three- or four-sided polygon by adding up the measurements of the sides where the measurements are either given or are determined by a previous direct measurement process. (See objective G20.)
4-6	**M31.** Improve the skill of estimation. By using the attributes introduced through these grade levels and the smaller units, the degree of accuracy of an estimate of a measurement should be fairly high. **M32.** Solve multi-step verbal arithmetic problems posed within a measurement context.	**M33.** Using a table of equivalents, solve simple problems involving the conversion of units within a system of measurement. **M34.** Use a formula to determine area of a square and a rectangle, and volume of a rectangular solid. By using addition, determine the perimeter of polygons of up to six sides. (See objective G21.)
7-8	**M35.** Determine area of triangles, surface area of rectangular solids and pyramids, and volume of cubes. Decompose complex planar figures into simpler ones. Use formulas as appropriate. (See objective G24.) **M36.** Use geometric relationships of complementary and supplementary angles, congruent and similar triangles, vertical angles, the angle sum of a triangle, and angles formed by parallels cut by a transversal to determine the size of angles in a figure when appropriate information is given. (See objective G25.) **M37.** Use the Pythagorean Theorem (and a calculator) to determine the length of missing sides in a right triangle. (See objective G26.)	**M38.** Make scale drawings. **M39.** Solve proportion problems posed within a measurement context. **M40.** Read and/or estimate distances on maps and globes. **M41.** Add, subtract, or multiply measurements (once known as denominate numbers) where regrouping may be required to determine answer. While this skill can be practiced in the abstract, the emphasis should lie in realistic problem situations arising out of active, direct measurement.
9-12	**M42.** Determine area of a circle, a trapezoid, and a parallelogram, surface area of prisms, cylinders, cones, and spheres, and volume of cylinders, prisms, pyramids, cones, and spheres. Use formulas as appropriate and/or as needed. (See objective G28.) **M43.** Use the trigonometry of right triangles to determine measures of sides and angles of triangles. **M44.** Use the properties of similar figures and the constant of proportionality to determine length, area, and volume measures of related figures. (See objective G31.)	**M45.** Determine the probability of simple and compound events, both independent and dependent. (See objective S14.) **M46.** Solve verbal arithmetic problems posed in a measurement context, including those that may require conversion of units.

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4

Geometry Overview

To think of geometry in terms of the logical development of Euclidean geometry as it occurs in a high school course is a much restricted and narrow point of view.

Geometry is the exploration and investigation of the various shapes we perceive in the world around us, of their properties, and of their mutual relationships. To think of geometry in terms of the logical development of Euclidean geometry as it occurs in a high school course is a much restricted and narrow point of view. One must not only consider the shape, size, and properties of one-, two-, and three-dimensional objects but also such notions as movement, transformations, position and location, orientation and incidence.

The primary goal of the teaching and learning of geometry is to develop knowledge and understanding of fundamental geometric objects and concepts. Familiarity and facility with fundamental geometry is basic to direct manipulation of the physical world, from carpentry to art to space flight, and to the representation of the physical world by arithmetic and algebraic models that permit its indirect manipulation and interaction.

Geometry is an important component of the mathematics curriculum throughout the elementary grades, and all students should complete at least one semester of geometry in high school.

Geometric Intuition

Included within this primary goal is the development of geometric intuition, especially regarding space, and the ability to call upon this intuition in using geometric modes of thinking in the solving of problems, in drawing geometric figures, and in making reasonable estimates of the expected outcome of geometric processes and calculations. Our world makes available a multitude of direct geometric experiences to assist and inspire this intuition: some seashells are symmetric; replacement spark plugs are congruent to the old ones; photo enlargements are similar to the original photo; the visual representation of synthesized sound is geometric; the outlines and designs on a balloon stay connected even when it gets blown up and stretched into different shapes.

Geometry and Other Subjects

Although one may espouse the study of geometry for its own sake because of cultural, aesthetic, or logical reasons, we should not lose sight of its interrelatedness with other branches of mathematics that are included in this guide. Although measurement theory and processes include more than geometric objects and their attributes, it is true that many examples of measurement and use of formulas are drawn from geometry. A major emphasis in statistics is on investigation of sets of data. The graphical display of such sets—scatter plots; box diagrams; line, bar, circle, and picto-graphs—is a vivid example of how the visual aspect of geometry helps to clarify a variety of concepts. Much of the study of relations and functions in algebra would be very difficult without their geometric representation on the coordinate plane. One-, two- and three-dimensional models such as the number line, rectangular arrays, and multi-base blocks serve to make arithmetic concepts and operations more clear. It is also possible to note the role of geometry in non-mathematical areas of thought. The vocabulary of geometric ideas such as *boundary, edge, lies on, is separated by* is also found in geography. Geometric forms are heavily used in non-verbal forms of communication as witnessed by the shapes used in traffic signs (e.g., the octagon for STOP or the circle to warn of a railroad crossing) and the simple, yet clever use of shapes in advertising logos used, for example, by automobile companies, television networks, or soft-drink manufacturers.

Problem Solving

Geometry also plays a role in problem solving. Problems that are geometric in nature can provide a welcome change of pace from problems that are strictly quantitative in nature. One of the major problem-solving processes recommended by authorities in the field is the drawing of a picture or graph to make relationships of a problem clear. This is a valuable use of geometry. When one looks for a pattern, often that pattern can be geometric or visual in nature. Again, this demonstrates the value of using geometric models in problem solving.

Geometry in the Early Grades

Instruction in geometry should be a set of experiences of "doing" mathematics. It is not necessary at the elementary level to force quantification into the activities in which children participate. Spatial relationships of plane and solid geometry should receive major attention. Experiences to develop these relationships should rely heavily on the use of manipulative devices as children explore, investigate, and discover the geometry of familiar objects. At a very young age, children should be taught the identification of two- and three-dimensional figures by handling objects. The processes of sorting and grouping can very easily lead to an awareness of the similarities and differences that exist among figures. Special properties of the figures can be specified at later grades as children learn to classify the objects. The concepts of similarity and congruence should be developed at this age through the same intuitive approach. Although the investigation of geometric concepts is to be carried out on an informal and exploratory basis, rudimentary skills in inductive and deductive reasoning can be developed at the upper elementary level. Brief but appropriate arguments presented by the teacher can provide a good foundation for later, more formal work.

Experiences to develop these relationships should rely heavily on the use of manipulative devices as children explore, investigate, and discover the geometry of familiar objects.

Coordinate Geometry

Coordinate geometry at the elementary grades should provide students with experience in locating points on a number line and the first quadrant of a two-dimensional coordinate grid. In the later grades all four quadrants can be included. Graphing of data should lead to exercises in problem solving and decision making. It is here that geometry plays a vital role as a visual tool for the representation and study of relations and functions.

Measurement

Measurement of geometric attributes is best done within the context of studying measurement as a process and a system. For a more complete discussion of this point, the reader should consult the Measurement section of this guide. It is sufficient to note here that excessive memorization of a large number of geometric measurement formulas is strongly discouraged.

The Beauty of Geometry

Applied and aesthetic uses of geometry should be obtained from various sources, including the environment, nature, art, industry, etc. At the secondary level, attention should be paid to the cultural significance of geometry as the oldest organized branch of mathematics. An inherent beauty exists in the visual and intellectual aspects of geometry and in its logical structure.

A De-emphasis of Proof in High School Geometry

... the teaching of proof and logical thought are not the main goals of geometry in high school....

In contrast to an older tradition, the teaching of proof and logical thought are not the main goals of geometry in high school, nor is geometry the only means of achieving those goals. The old view of the study of geometry in high school as primarily a vehicle for teaching the nature of a mathematical system (axioms, definitions, logical rules, and theorems) arose when geometry was the only such system available. In fact, detailed knowledge of mathematics as a system is not necessary at the high school level, and at any level simpler systems such as group theory and finite geometries provide an adequate introduction. As in the rest of mathematics, sound arguments and the correct use of logical principles should be stressed. Nevertheless, formal "two column" proofs should be deemphasized; they are not used elsewhere. The time gained by the deemphasis of formal systems and proof, can be used for the initial study of algebraic and vector methods in geometry, and the experiential development of geometric intuition.

It would be appropriate, however, for programs to treat deductive systems in advanced eleventh and twelfth grade courses. If so, varieties of forms of proof should be addressed in a variety of mathematical contexts (algebra, trigonometry, geometry, number theory, probability).

Transformational Geometry

Because traditional Euclidean geometry at the secondary level is entirely static, the dynamic aspects of geometry (for example rigid motions and vectors) have often been neglected. Geometric interrelations are often clarified by a dynamic approach. Transformational geometry and its study of rotations, reflections, translations, and dilations provide a set of experiences that illustrate this dynamic nature.

Geometric interrelations are often clarified by a dynamic approach.

Technology

Technology provides an excellent new resource to develop many concepts in geometry in exciting and stimulating ways. The latest computer hardware that provides innovative interactive methods with the students should be integrated into the study of geometry. A language such as Logo can provide students with experiences in geometry and problem solving that have not been possible up to this time. The motivation for the student to use technology can be a great catalyst for learning.

Enrichment

For mathematically talented students it is appropriate to include the following topics: trigonometry of the right triangle; coordinate geometry of lines, planes, conic sections, quadric surfaces, and simple functions; elementary properties of convex sets and polyhedra; transformation geometry; vector geometry; an introduction to the geometry of linear algebra and linear programming; non-Euclidean geometry; topology.

Table 3

Geometry Objectives

Figures, Properties, and Relationships	
K-3	**G1.** Explore familiar two- and three-dimensional objects. Properties such as inside, outside, flat, curved, straight, round, square, large, and small should be included. **G2.** Identify and classify the following figures through visual observation and identified properties: triangles, squares, rectangles, cubes, spheres, circles, open and closed curves. **G3.** Identify perpendicular and parallel lines. **G4.** Explore simple patterns of symmetry in the environment and the natural world. **G5.** Explore the relationships of similarity and congruence.
4-6	**G6.** Identify and classify the following figures and their parts: rectangular prism, cylinder, cone, pyramid, parallelogram, quadrilateral, isosceles triangle, angle (right, acute, and obtuse), circle, rectangle, rhombus, square, and trapezoid. These abilities include being able to state the name when given a physical or pictorial model and the reverse, to produce a physical or pictorial model when given the name. **G7.** Understand that lines, line segments, rays, angles and space can be considered as sets of points. Draw, describe or identify relationships such as perpendicularity, parallelism, and incidence relations such as intersecting, between, in the interior of, on, in the exterior of, on opposite sides of. **G8.** Demonstrate an understanding of symmetry by using more complex patterns and models than those used in Grades K-3. Examples are paper folding, mirror images and pattern drawings on graph paper. **G9.** Identify congruent and similar figures and verbally describe some of the properties of such figures and the relationships among corresponding parts.
7-8	**G10.** Identify and classify figures, their properties, and relationships among the parts of a figure and between two figures. In addition to the figures listed in earlier grades, adjacent angles, vertical angles, supplementary and complementary angles should be included. Emphasis should be placed on parts of a figure such as legs and hypotenuse of a right triangle, the diagonal of a rectangle, the radius and diameter of a circle, etc. These abilities include being able to state the name when given a physical or pictorial model or given a verbal description or definition, and the reverse, being able to produce a physical or pictorial model or give a verbal description or definition when given the name. **G11.** Create patterns that demonstrate an understanding of the concept of reflectional or rotational symmetry in two- and three-dimensions. **G12.** Construct oral and/or written arguments that demonstrate an understanding of why two figures are congruent or similar; or why two lines or planes are perpendicular or parallel. **G13.** Acquire the concepts used in the transformations of reflecting, rotating, and translating.
9-12	**G14.** Identify and classify the following figures or parts of figures: segment, ray, angle, polygon, diameter, chord, tangent, arc, polyhedron, convex region. These abilities include being able to state the name when given a physical or pictorial model or given a verbal description or definition and the reverse, being able to produce a physical or pictorial model or give a verbal description or definition when given the name. **G15.** Describe or define the term when given the name, or a physical or pictorial example, of: lines, segments, and rays – parallel, perpendicular, skew, slope, intersecting planes – parallel, perpendicular, intersecting triangles – median, angle bisector, altitude, isosceles, equilateral, right, acute, obtuse angles – acute, right, obtuse, exterior (of a triangle), adjacent, complement, supplement, vertical, inscribed and central (with respect to circles) polygons – isosceles, equilateral, regular, convex, diagonals, vertices relations – congruence, similarity, symmetry

Figures, Properties, and Relationships (continued)

9-12 (cont.)

G16. State the familiar relationships and properties of the following figures and their parts: angles formed by two intersecting lines; a triangle and the sum of the measures of its angles; isosceles triangle; a triangle and its exterior angles; parallel lines cut by a transversal; diagonals of parallelograms, rectangles, rhombuses, and squares; medians, angle bisectors, and altitudes of triangles; interior and exterior angle sum of regular polygons; circles, radii, and tangents, equal chords, arcs, central angles; points, lines, and planes and intersections of these in various combinations; intersection of a plane with cones, spheres, and cylinders.

G17. Use definitions, assumptions, and theorems related to congruence and similarity to establish simple results about polygons and circles.

G18. Distinguish between a statement and its converse, and recognize that the two are not logically equivalent.

Measurement and Construction

K-3

G19. Become familiar with attribute of length. By using either arbitrary or standard units, students should experience the direct measurement of this attribute. Use a ruler to draw a segment of a given size, correct to the nearest half-inch or centimeter. (See objectives M2, M14)

4-6

G20. Become familiar with attributes of angle measure, area, perimeter, and volume. Using a protractor, directly measure angles to the nearest degree and draw angles of a given size. Directly measure or draw segments to the nearest sixteenth of an inch or millimeter. (See objectives M4, M19, M23, M24, and M30.)

G21. Use a formula to determine area of a square and a rectangle, and volume of a rectangular solid. By using addition, determine the perimeter of a polygon of up to six sides. (See objective M34.)

G22. Use appropriate construction instruments (e.g., compass, straight-edge, etc.) to explore simple geometric properties and concepts.

7-8

G23. Extend earlier notions of perimeter and area to include the specialized attributes of circumference and surface area. Experimentally determine an approximate value of π (Pi) as a constant associated with a circle. (See objective M8.)

G24. Determine area of triangles, surface area of rectangular solids and pyramids, and volume of cubes. Decompose complex planar figures into simpler ones. Use formulas as appropriate. (See objective M35.)

G25. Use geometric relationships of complementary and supplementary angles, congruent and similar triangles, vertical angles, the angle sum of a triangle, and angles formed by parallels cut by a transversal to determine size of angles in a figure when appropriate information is given. (See objective M36.)

G26. Understand and use the Pythagorean theorem (and a calculator) to determine the length of missing sides in a right triangle. (See objective M9 and M37.)

G27. Use basic instruments (straight-edge, compass) to construct congruent segments and angles, bisectors of segments and angles, perpendicular lines, perpendicular bisectors, and parallel lines. This is not just a skill objective, but is intended to aid in conceptual development and understanding of various properties and relationships.

9-12

G28. Determine the area of a circle, a trapezoid, and a parallelogram, the surface area of prisms, cylinders, cones and spheres, and the volume of cylinders, prisms, pyramids, cones, and spheres. Use formulas as appropriate. (See objective M42.)

G29. When given appropriate measures, use known relationships to determine the size of missing parts of figures. For example, determine the size of a base angle of an isosceles triangle when the size of the vertex angle is given, or use similar triangles and ratio and proportion to find the length of a side of a triangle.

Table 3 *(continued)*

Geometry Objectives

Measurement and Construction (continued)		
9-12 *(cont.)*	**G30.** Use appropriate tools to copy angles, segments, and triangles to construct simple regular polygons, to divide a segment into equal parts, and to draw a circle through three non-collinear points.	**G31.** Understand and use the effects of similarity on perimeter, area, volume, and surface area. (See objective M44.)
Coordinate Geometry†		
K-3	**G32.** Identify points representing positive whole numbers on the number line. (See objective A31.)	**G33.** Explore the use of coordinates to find locations in the first quadrant.
4-6	**G34.** Use concepts related to the coordinate plane such as map reading and graphing. Included would be activities leading to the identification of points (using appropriate terminology) in the first quadrant.	
7-8	**G35.** Graph solution sets of number sentences (equations and inequalities) on the number line and coordinate plane. (See objective A35.)	
9-12	**G36.** Associate a point on a line, plane, or space with its coordinate(s) and conversely. **G37.** Determine the length of a segment on a line or on a plane or •in space. **G38.** Determine the coordinates of the midpoint of a segment on a line or on a plane or • in space. **G39.** Given its equation, graph a line, and understand the algebraic meaning of the point(s) of intersection of such graphs. (See objective A38.) **•G40.** Given two points on a line or one point and the slope of the line, write the equation of a line on a plane. (See objective A38.)	**•G41.** When given a graph or equation of a line in a plane, give the slope and intercepts of that line. When given their equations, state whether two lines on a plane are parallel or perpendicular. (See objective A38.) **G42.** Associate equations with graphs of circles, and of •simple parabolas, ellipses, and hyperbolas, and graph these curves from their equations. **G43.** Sketch the graph of a simple function or relation of one variable, and •use the graph and its symmetries to describe the properties of the function or relation. **G44.** Graph linear inequalities and •identify the convex region associated with a system of such inequalities. • *Indicates an objective that is fully developed only in courses for college-intending students.*

† *See also* Visual Exploration of Data *in the Statistics strand, p. 49, and* Graphs *in the Algebra strand, p. 60.*

Problem Solving

K-3	**G45.** Investigate properties of and relationships among geometric figures. Included here would be making and testing conjectures, as, for example, what would happen if a square were cut into two pieces along a diagonal.	**G46.** Use organized lists and diagrams to show properties of and relationships among geometric figures. **G47.** Relate physical world examples to the ideas and concepts of geometry.
4-6	**G45.** Investigate properties of and relationships among geometric figures. Included here would be making and testing conjectures, as, for example, what would happen if a square were cut into two pieces along a diagonal.‡	**G46.** Use organized lists and diagrams to show properties of and relationships among geometric figures.‡ **G47.** Relate physical world examples to the ideas and concepts of geometry.‡ ‡ *The degree of sophistication and complexity of situations and examples increases with grade levels.*
7-8	**G48.** Develop ways to verify that two figures are similar or congruent. This may include construction or a logical argument. (See related objectives G12 and G22.) **G49.** Collect and use information to draw conclusions about properties of geometric figures which in turn could lead to the classification of the figure. **G50.** Construct an organized classification scheme for space figures and make graphic representations of that scheme (for example, a Venn diagram).	**G51.** Use valid deductive reasoning with geometric situations and use inductive reasoning to formulate generalizations about geometric patterns. **G52.** Relate physical world examples to the ideas and concepts of geometry acquired at these grade levels. Recognize when such a geometric modeling of a physical situation would aid in the solution of a problem.
9-12	**G53.** Use the simple laws of logic, including indirect reasoning, to determine whether a result has been validly concluded. **G54.** Solve purely geometric problems requiring application of more than one relationship or skill. •**G55.** Use geometric representations to solve problems in related fields. Use vectors for directed magnitudes or complex numbers, and draw graphs of algebraic equations and inequalities.	**G56.** Use geometric relationships to solve problems in applied fields such as navigation, landscape architecture, carpentry, dressmaking, computer graphics, and commercial art. • *Indicates an objective that is fully developed only in courses for college-intending students.*

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Statistics

5

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Statistics Overview

America today is in the information age. Data collecting and processing are an integral part of industry, business, government, politics, advertising, and entertainment. People are confronted daily with data from which they must make decisions on health, employment, parenting, or finances. In *An Agenda for Action: Recommendations for School Mathematics of the 1980s*, the National Council of Teachers of Mathematics recommended that the curriculum for the eighties include "locating and processing quantitative information; collecting data; organizing and presenting data; interpreting data; drawing inferences and predicting from data." The National Science Board Committee on Precollege Education in Mathematics, Science and Technology stated that the elementary schools should provide "basic understanding of data analysis, simple statistics and probability" and that ". . . elementary statistics and probability should now be considered fundamental for all high school students."

These reports are typical. Schools are being asked to produce students who can effectively cope with the vast amounts of data they will encounter in almost every facet of their environment. Thus, statistics, the study of data, must become a vital component of mathematics instruction, integrated into the entire K-12 mathematics curriculum.

. . . statistics, the study of data, must become a vital component of mathematics instruction, integrated into the entire K-12 mathematics curriculum.

Statistics in Society

The role of statistics in society should provide the framework for the entire study of statistics. Students must recognize how data and the interpretation of these data affect their lives. Many everyday decisions are made on the basis of past experience with quantitative information. There is a chance of being right and of being wrong. Statistics is used to minimize this error. Students should be exposed to statistical arguments and be able to determine if these arguments are valid. Students can find examples of misused statistical techniques within their own experiences. They should be able to look at numerical data from different areas and to say what these data mean.

At one time the traditional statistics course focused on the use of a set of statistics to describe the parameters of a population and to accept or reject a hypothesis in regard to that population. A variety of formulas and tests were prescribed for a variety of situations. Often a student would work on a textbook problem to arrive at a contrived conclusion. In the present information age, however, things have changed. Statistically literate citizens must understand and be critical of the statistical information found everywhere around them. And every citizen should be statistically literate.

To achieve the desired level of statistical literacy, the task force recommends instruction in statistics in all elementary grades and a semester of statistics for all students in high school. This one semester of statistics could be offered as a discrete course, but would be more appropriately integrated into all course work at all grade levels.

Exploratory Data Analysis

This new emphasis on statistics together with the vast amount of data being generated have created a need for new techniques to organize and interpret these data. Exploratory Data Analysis, as these new techniques are called, should become a prerequisite to traditional statistics programs and provide for all students the basic skills necessary to deal with data in today's information age. The rigid analyses of the traditional statistics course can be saved for those few who need such a course.

Today the study of statistics centers on the concept of data. Students must learn to analyze the reliability of a set of data and to organize a set of data to most effectively interpret that data in a meaningful way. The data used by students should be based on actual experiments and related to real-world experiences. Newspapers, magazines, almanacs, sports encyclopedias, record books, texts, and advertising brochures are excellent sources and can be relevant to students at every grade level.

Sampling and Randomness

Students should be introduced to the notions of sampling and randomness. Consideration should be given to why samples are taken, who takes samples, and how they are taken. The impact of bias on a set of data should be included. The student should be aware of the relation of a sample to the population and recognize that the very nature of sampling introduces a variability factor into the data. The results of the analysis will reflect the degree of this variability or "fuzziness."

Data Interpretation

Students must learn to evaluate the quality of a set of data in terms of its source, method of collection, completeness, correctness of calculations, absence of bias, and whether it was drawn from a valid sample. Once data have been collected, the students must learn to organize and to make interpretations from these data. In exploratory data analysis, the techniques for organizing data are primarily graphic, using new and often nontraditional displays. Students no longer make only bar graphs, line graphs and circle graphs but construct stem and leaf plots, box plots or QQ plots. These provide a basis from which to interpret the data. The purpose of the graph is communication. The focus is not on each individual piece of information but on what that information collectively reveals. The user of statistics should learn to identify trends, high and low points, relative spread, continuity, concavity, gaps, modality, increasing or decreasing intervals and outliers. Again, students should recognize that visual displays are subject to a certain amount of variability, variability that is inherent in the very source of the data.

In exploratory data analysis, the techniques for organizing data are primarily graphic, using new and often nontraditional displays.

Probability

Probability should be treated primarily as a tool necessary to work with data and to make predictions from that data. Probability is a measure of the uncertainty of an event, a measure which changes as more information is gained. The student should know the difference between theoretical probability and experimental probability and understand that the probability of any set of events is subject to certain axioms. The probabilistic relation between a sample and a population should be intuitively presented and explored. Simple models such as dice, coins, and spinners will reinforce the concepts. Experiments should be easily managed by all students. The objective is to establish certain basic probability concepts and not to explore some of the more complex probability problems. These often seem fascinating but should be saved for a more in-depth treatment in an advanced course.

Interdisciplinary Nature of Statistics

If students are to recognize the importance of statistics in society, they must begin by recognizing the interdisciplinary nature of statistics in their schools. Because data are an integral part of many fields, statistics is a tool which should be used in many disciplines such as science, psychology, social studies, or economics. Communication courses can involve the results of data analysis, reporting surveys, or using charts to verify or illustrate points. When teaching statistics, every effort should be made to relate concepts to these other disciplines. Problems should be taken from other class settings and should be solved cooperatively with other teachers. In order to obtain the best possible foundation for understanding work with statistics, students at every level should prepare an individual project. These projects, if appropriate for each ability level and interest area, can and should exemplify statistics in society. The student projects should be based in subject areas other than mathematics and assigned in conjunction with work in other classes. Statistics should not be rotely learned from a mathematics text and discussed only in mathematics time.

When teaching statistics, every effort should be made to relate concepts to . . . other disciplines.

Problem Solving

Problem-solving strategies are inherent in exploratory data analysis. The use of diagrams, tables, and charts is the primary method for organizing and displaying data. Looking for patterns, predicting trends, making organized lists are standard procedures for drawing conclusions from data. Models are used to simulate situations to determine probabilities or to represent certain sets of data. In society, problems rarely have a unique answer. Problems in statistics, unlike most applications in mathematics, provide students with exposure to this problem-solving concept. There is no one correct way to organize and display data and there is no one right interpretation.

Calculators and Computers

The calculator is an essential tool in working with data. In too many cases, concepts can be obscured because of the complexity of the arithmetic. Understanding the significance of standard deviation is more important than the ability to calculate it correctly. While a computer can enhance the learning of statistics, it must be used cautiously. A student's initial experience with data should not be gained from machine-generated data but from actual student-gathered information. Initial sampling techniques should also be investigated with "real data." If too much use is made of random sets of data generated by a computer too soon in the learning process, students will lose sight of the fact that statistics deals with actual information and is not merely a science of numbers. Computers can, however, be valuable tools in organizing data and in providing a variety of displays for data. Experiments can easily be simulated on a computer, but students must first understand the process of simulation and be able to interpret the results.

The calculator is an essential tool in working with data.

Enrichment

Statistics provides many opportunities for the mathematically talented student. Each topic has the potential for several levels of difficulty. Simple, direct manipulations and graphs can be used for a given data set. The same set can be presented in a variety of sophisticated displays which can challenge the imagination and creativity of any student. Simple patterns can be discerned by all students. More subtle inferences may be made by those with more insight. Student projects can be extended to challenge those of every ability level. The talented student can use exploratory data techniques as a basis for the more traditional and rigorous procedures. A set of enrichment objectives follows in table 5, p. 51.

Table 4

Statistics Objectives

Statistics in Society	
4-6	**S1.** Explore the importance of statistics in society. **S2.** Become familiar with the use of numbers and graphs in newspapers, magazines, television, and other sources within society.
7-8	**S3.** Become familiar with the United States Census. Know how the census is given, when it is given, what kind of questions are contained on the census, why the census is given, and what uses are made of the information gathered. **S4.** Recognize the use and misuse of statistics in newspapers, magazines, texts, television, and other sources in society.
9-12	**S5.** Apply the methods of statistics to study situations drawn from such areas as politics, advertising, medicine, business, industry, and science. The question of the role of ethics in applications of statistics should be addressed.
Probability	
K-3	**S6.** Explore the concept of chance, based upon repeated observations of real world events such as weather, games, or contests.
4-6	**S7.** Explore the concept of probability by working with simple models such as dice and coins and also by conducting simple experiments to make probability predictions.
7-8	**S8.** Estimate the probability of a given event from experimental data and calculate theoretical probabilities. Know the probability of an impossible event is 0 and of a certain event is 1. The situations used as models and for experimentation should be simple and within a familiar environment. **S9.** Know the probability of a given event is a fraction between 0 and 1 and that the sum of the probabilities of the outcomes of an experiment is 1. Recognize when the outcomes of an experiment are equally likely and the effect this can have on the probability. **S10.** Using tables and graphs, explore the concept of a probability distribution. **S11.** Explore concepts of independent, dependent, and mutually exclusive events. **S12.** Compute the odds for or against a given event.
9-12	**S13.** Compute the total number of outcomes of compound experiments by using the counting principle or by constructing a tree diagram. **S14.** Determine the probability of simple and compound events, both independent and dependent, and of the complement of an event. (See objective M45.) **S15.** Compute the expected value of an event. **S16.** Express the total number of outcomes of an experiment as percents and graph the results. Identify this graph as a probability distribution and relate the area associated with a given outcome to the probability of that outcome. **S17.** Use simulations to generate sampling distributions. Recognize the difference between the graph of a discrete distribution and a continuous distribution. Identify a distribution that is approximately normal. This includes the ability to recognize the common characteristics of a normal distribution such as bell shaped symmetric, and having concurrent mean, median, and mode.

Collecting Data

K-3	**S18.** Gather data by counting, by performing simple experiments, by measuring, or from various media sources such as newspapers, magazines, almanacs.
4-6	**S19.** Explore kinds of data generated by others and evaluate the sources of that data. Recognize that bias in a question can affect results or interpretation of that data. **S20.** Gather data by conducting a survey, or by carrying out a simulation.
7-8	**S21.** Recognize that the quality of a set of data depends on the source of a set of data and on the gathering techniques used. **S22.** Distinguish between a survey (sample) and a census (population), and understand when each is necessary, why a survey might be given, and by whom.
9-12	**S23.** Explore the concept of randomness. Use a random number table and a set of random numbers generated by a computer. **S24.** Formulate appropriate questions for a given objective, and construct an experiment to meet that objective. This includes identifying a suitable population and deciding if a survey or census is appropriate. If a survey is chosen, select an appropriate sampling method. The issue of bias in the selection of the sample and possible bias due to nonresponse should be addressed. Recognize the variability of different samples drawn from the same population. Define a random sample and distinguish those which are random from those which are not.

Visual Exploration of Data†

K-3	**S25.** Organize a set of data by tallying and ordering. **S26.** Construct bar graphs, histograms, and pictograms. State impressions obtained from these graphs. The data for these visual displays should come from realistic problem situations that occur within the classroom. (See objective A30.)
4-6	**S27.** Construct a frequency distribution for a given set of data. **S28.** Construct simple line graphs and stem and leaf plots. State impressions obtained from such graphs, constructed either by the student or by another person. **S29.** Using a graphic representation of a set of data, recognize and describe patterns such as increases, decreases, or trends.
7-8	**S30.** Construct scatterplots, circle graphs, frequency polygons, and box plots. State impressions obtained from those graphs. **S31.** Identify trends or variations in the graphic representations of a given set of data. This includes finding the deviant behavior of a specific point, finding breaks in patterns, and looking at outliers. **S32.** Convert raw data into grouped data when necessary and graph the results. **S33.** Construct a median fit line and then use this line to make predictions and to fill in information missing from the graph. **S34.** Develop the concept of correlation. Identify correlation as positive, negative, or 0 for a given scatterplot. Recognize the distinction between correlation and causation. **S35.** Compare sets of data represented graphically.

†*See also* Graphs *in the Algebra strand, p. 60, and* Coordinate Geometry *in the Geometry strand, p. 40.*

Table 4 *(continued)*

Statistics Objectives

Visual Exploration of Data (continued)		
9-12	**S36.** Explore different methods of charting a given set of data. Recognize that different organizations of the same data may yield additional information. **S37.** Select appropriate graphing techniques to represent a set of data.	**S38.** Construct a cumulative frequency polygon for a set of data. State impressions obtained from that graph. **S39.** Recognize the variation inherent in the graphic representation of a set of data and be aware of how this variation can affect any conclusions.
Descriptive Statistics		
K-3	**S40.** Find the mode(s) and range for a given set of data.	
4-6	**S41.** Find a measure to describe the central tendency of a set of data (mean, median, or mode(s)), and determine if the result is reasonable. The data should be simple and limited in quantity. Use a calculator to deal with realistic sets of data.	
7-8	**S42.** Define percentile and quartile and determine a specific percentile or quartile. This includes the ability to distinguish between percent and percentile.	**S43.** Determine the mean, median, and mode(s) from data displayed in a frequency distribution. Understand the difference in using each of these as a measure of central tendency.
9-12	**S44.** Describe the dispersion of a set of data.	
Inferential Statistics		
9-12	**S45.** Recognize valid and invalid reasoning. Evaluation should be made of statistical arguments found in newspapers, on television, in magazines, and other sources relevant to the student.	**S46.** Express conclusions and interpretations in written form for a given representation of data. The ability to clearly communicate the results of an analysis of a set of data should be considered a part of the procedure.

Table 5

Statistics Enrichment Objectives Grades 9-12

Probability

E1. Distinguish a permutation from a combination. Compute the number of outcomes of an experiment when the computation requires the use of permutations or combinations.

E2. Calculate the probability of the intersection and of the union of two or more events.

E3. Compute the conditional probability of an event.

E4. Apply Bayes's formula to calculate the probability of an event.

Collecting Data

E5. Select an appropriate sampling method for a given experiment. Establish more complicated experiments when investigating a given topic.

E6. Use the techniques of experimental design to gather data for a given objective. a measure of central tendency.

Descriptive Statistics

E7. Compute the variance for a given set of data.

E8. Compute the standard deviation for a given set of data, using more complicated numbers and large data sets.

E9. Compute the correlation coefficient for a given set of data.

E10. Compute the rank correlation for a given set of data.

E11. Write a computer program to compute any of the descriptive statistics covered.

Probability Distributions

E12. Compute the mean and standard deviation of a binomial distribution given the number of trials and the probability of success. Determine theoretical probability from a binomial distribution.

E13. Standardize a normal distribution for a given mean and standard deviation. This includes the ability to use a table to estimate the probability of an event that is normally distributed.

E14. Recognize the characteristics and applications of the Poisson distribution.

E15. Compute the expected value of a random variable given its probability distribution.

Inferential Statistics

E16. Recognize when a rank correlation test is appropriate and apply it correctly.

E17. Compute a confidence interval and apply the concept effectively in a decision-making situation.

E18. Estimate population parameters from sample statistics.

E19. Design and complete an appropriate hypothesis test for a given situation.

E20. Recognize the appropriate test to use for a given set of data and apply it correctly. This includes using the chi square test, Z (standard normal) test, and the t test.

E21. Construct a linear regression line given paired data and determine if a given regression line is a good fit for the data.

E22. Recognize when a rank sum test is appropriate and apply it correctly.

E23. Recognize when nonparametric methods are appropriate for a given set of data.

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6

Algebra Overview

In its 1980 report, *An Agenda for Action*, the National Council of Teachers of Mathematics asserts as its major recommendation that "Mathematics programs of the 1980s be designed to equip students with the mathematical methods that support the full range of problem solving." The *Agenda* goes on to suggest that the notion of function and mathematical symbolism be used to describe real world relationships. "Students must (also) learn to . . . discover patterns and similarities . . . ." The Algebra strand addresses these concepts.

Algebra at All Grade Levels

Algebra and algebraic thinking and methods need not be restricted to the traditional high school course(s). From the day children enter school, teachers should organize experiences that will encourage children to use higher order thinking processes, to seek and discover ideas for themselves, to look for patterns and relationships, and to express generalizations. The Conference Board of the Mathematical Sciences issued a report in 1982 entitled *The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not* in which it suggested for the elementary and middle school grades that "Function concepts including dynamic models of increasing or decreasing phenomena should be taught" and that ". . . Based upon motivation from arithmetic, algebraic symbolism and techniques should be encouraged, particularly in grades 7 and 8."

Because of its importance in giving students methods of organizing and understanding observations of the world around us, it is recommended that *all* students study the equivalent of one full year of algebra during high school. The content and degree of emphasis will vary according to individual abilities and career goals, but the major objectives for high school, and earlier grades as well, should be centered on three major topics: language and symbolism, relations and functions, and graphs. Algebra offers a unique opportunity to develop linkages among the various parts of mathematics which sometimes appear to be unrelated topics isolated in discrete courses such as algebra, geometry, statistics, and calculus. The use of variables, functions, relations, and graphs are among the prime tools of problem solving, and constitute a common core in the mathematics curriculum K-12.

The use of variables, functions, relations, and graphs are among the prime tools of problem solving, and constitute a common core in the mathematics curriculum K-12.

Language and Symbolism

A major component in the language of algebra is the use of variables to represent numerical quantities. The young child may encounter this concept in finding a replacement for a "box" in an open sentence expressing a basic fact of addition. Experience in grades K-3 should also include real world situations that motivate sentences such as $\square + 5 = 7$;

$17 - \square = 8$; and $\square - 8 = 4$ as well as the more familiar sentences where a sum or difference is to be found. Strategies for "solving" these open sentences should include the use of manipulatives, tallying, counting, and finding related facts. Transforming the sentence to isolate the variable is far too complicated for the early grades.

In later grades, literal variables may be used in expressing common formulas for calculating area or volume, and methods of solution can expand along with the size of the numbers and the scope of applications. The use of mathematics to organize, to look for patterns, to develop generalizations, and to solve problems requires an understanding and use of variables in which the pupil uses the language of mathematics to express real-world situations. In a very real sense this involves the pupil in a creative act. It also strengthens the bond with the concrete involvement that is used to develop abstract mathematical concepts.

Relations

Most of mathematics is concerned with relations. The young child learns early to relate certain objects or sounds with other objects. For example, a child associates other children with their names. Intuitively, a child recognizes certain associations between pairs of objects or names of objects. Thus the child learns to form ordered pairs, such as name and object, in the development of language. The process of forming pairs should be introduced early in the mathematics program. In beginning arithmetic, pupils find that counting is a way of determining what number is to be associated with a collection of objects. Thus, counting determines certain related pairs; namely, (set, number).

The relationship "less than" is another example of related pairs. It has the form (number, greater number). This is an example of a relation in which a single number is related to many other numbers. As pupils develop skill in collecting empirical data, they should begin to search for meaningful relations in the data. For example, a pupil could discover the relation between the number of sides of a polygon and the number of diagonals that can be drawn.

Functions

One type of relation of particular importance to science and mathematics is called a functional relation or, simply, a function. Functions have just one related pair which exists with a given first member. Some functional notation should be used systematically by the end of the eighth grade. Several notational schemes should be used on occasion since different notations are suggestive of different aspects of the function concept.

Some functional notation should be used systematically by the end of the eighth grade.

The function concept includes mathematical operations. Elementary pupils encounter functions when learning the number facts. Multiplication by five, for example, identifies a function defined on the set of numbers. Pupils should realize that functions can be identified by statements, formulas or equations, tables of data, and graphs. The mathematics program should acquaint the pupil with functional notation and should develop the pupil's ability to plot linear and quadratic functions as well as functions with "jumps," such as the greatest integer function.

Graphs

Learning to use graphs to present information is essential, particularly in the study and use of statistics. Graphs provide pupils with an organized method of recording and communicating their observations. Observations can be recorded as soon as a child has reached the collecting stage. It is while collecting and sorting data that the pupil begins to make comparisons and form relations.

Use of a pictorial description of pairs of objects should begin early in the mathematics program. A class could graph the height of a plant in successive weeks, or record temperatures for each day in the month. The pupil's experiences with such graphing reinforces the concept of the number plane, presents a picture for linear relations, and provides an excellent way to develop an intuitive understanding of the concept of relations.

Graphing is also invaluable in the applications of mathematics. A class may record the length of a spring or rubber band as weights of successive sizes are suspended; or at a more advanced stage, the period of a simple pendulum could be recorded while the length of the string is varied. In the study of measurement, a class may record the amount of water required to fill cylindrical jars of various diameters to a fixed depth. In the study of geometry, a class may plot circumference as related to diameter for different circles.

The mathematics program should include the study of the coordinate plane. The association between each point in the plane with an ordered pair of numbers is basic to the mathematics that connects geometry and algebra. This association is also central to the understanding of maps and, more generally, of scale drawings.

The association between each point in the plane with an ordered pair of numbers is basic to the mathematics that connects geometry and algebra.

Problem Solving

Problem solving is both a goal and a major activity for objectives in algebra. This strong interrelationship can be seen in each of the three sub-topics.

Language and Symbolism

A powerful problem-solving strategy is to represent the problem situation in mathematical language. Experiences in expressing relationships using variables, operations, and numerals will contribute to students' problem-solving abilities. Activities to motivate, develop, and apply the use of variables should include a wide variety of problem-solving situations.

Because there is so much reliance on language and symbolism throughout all of algebra, there is a great deal of overlap between this sub-topic and the other sub-topics. Here the emphasis is upon the process of representation by algebraic symbolism; in the other sub-topics the emphasis is upon what is being represented.

Relations and Functions

Another problem-solving strategy involves an investigation of simpler versions of the problem or special cases. If these studies reveal a pattern, the key to solving the original problem may emerge as an extension of that pattern. An awareness of patterns and tools for detecting and extending patterns is among the goals of algebra.

Graphs

The usefulness of graphing in problem solving can span the entire process. Graphs can facilitate "good" guesses for early exploration and investigation. Graphs can yield insights about relationships or patterns and provide good reference points with which to judge the reasonableness of an answer.

Problem-solving strategies can be seen at two levels in the goals of algebra. On the one hand, mathematical functions are at the root of many problem situations—from simple functions like direct variation to trigonometric, logarithmic, or exponential functions. On the other hand, the methods of studying those functions frequently involve the creation and/or interpretation of tables of data—in their own right very useful problem-solving strategies.

Calculators and Computers

The emerging use of calculators and computers in the classroom offers both a motivation and a context for the development of objectives in algebra. The concept of variable is closely related to the notion of naming storage locations in computer programming. The use of ordered pairs is an important part of storing, searching, and sorting elements of two dimensional arrays of data. Most computers and many calculators have built-in functions (such as sin, cos, log, sqr, sum, and int) which allow for easy access to derived values. Use of these devices can assist in developing the concepts underlying these functions. The graphics capabilities of microcomputers should enable teachers to provide highly detailed demonstrations of the behavior of special functions, to illustrate concepts such as minimum, maximum, roots, common values, and limits, to explore the effects of transformations on functions, and to develop in a dynamic manner topics such as area under a curve.

The emerging use of calculators and computers in the classroom offers both a motivation and a context for the development of objectives in algebra.

Technology can be a highly effective classroom tool for concept development, for practice of skills and concepts, for applications of those skills and concepts, and for the development of problem-solving strategies. The topics of algebra lend themselves particularly well to all four uses. To fully exploit the power of this technology, the students need to take an active role in using the calculator and the computer as tools of mathematics.

Enrichment

The mathematically talented student should be encouraged and challenged to explore and develop the unifying concepts of function and relation beyond the levels expected of average students. This can take the form of studying different types of functions in greater depth (for example, finding inverse functions, finding minimums, maximums, and critical values) or studying functions which are not normally covered at that level, such as exploring the Fibonacci sequence, its relation to nature, and some of the hidden relationships it contains.

Algebra Objectives

Language and Symbolism

K-3	**A1.** Explore the concept of the "unknown" using physical objects. (How many are in the box? Take three away. How many are in the box now?) **A2.** Represent active (joining or separating) or static (comparison, part-whole) situations by using numbers, operations (+, −), and relations (=, <, >) which may or may not include an unknown.	**A3.** Use numbers to replace a box representing an unknown quantity in a number sentence (for example, 9 + □ = 14) and determine if the replacement makes the sentence true. Because the concern is for the replacement (substitution) process and not on memorization of number facts, a calculator would be an appropriate tool to use.
4-6	**A4.** Represent problem situations by using numbers, operations (+, −, ×, ÷), and relations (=, <, >) which may or may not include an unknown.	**A5.** Evaluate simple expressions involving variables. The expression could be a familiar formula such as $A = l \times w$ or an algebraic one such as $2n - 3$.
7-8	**A6.** Represent problem situations by using expressions involving variables.	**A7.** Understand the role of a variable in the statement of a general principle (for example, the commutative property of addition).
9-12	**A8.** Evaluate expressions involving variables by using numerical substitution. These expressions would be more complex than in earlier grades. Evaluation involves knowledge of grouping symbols, order of operations, and exponential notation. **A9.** Simplify arithmetic and algebraic expressions by combining like terms. **A10.** Simplify polynomials by factoring out common monomial factors.	**A11.** Represent problem situations by using variables, equations, inequalities, and systems. The situations would be more complex than those encountered in earlier grades. ***A12.** Simplify algebraic expressions involving trinomial factoring and special products. ***A13.** Simplify algebraic expressions including simple rational and radical expressions. **Indicates an objective that is fully developed only in courses for college-intending students.*

Relations and Functions

K-3	**A14.** Develop pattern recognition. This would be carried out primarily by working with simple sequences that are determined by numerical or geometric properties, or by other attributes such as color or orientation. The sequences could be represented either physically, as with pattern blocks, or pictorially or symbolically. If enough elements are presented, the child would describe the rule or relation that determines the sequence and then continue the sequence.	**A15.** Explore the concept of ordered pairs by linking such things as objects and names. Included would be the development of the skill of finding the second element of an ordered pair given a list, picture, or rule that characterizes the relation.
4-6	**A16.** Solve an open sentence written either with a "box" or a literal variable to represent the unknown by using a guess and test procedure.	

Table 6 *(continued)*

Algebra Objectives

Relations and Functions (continued)

7-8	**A17.** Represent a relation given by a table of input and output numbers by writing a mathematical sentence. **A18.** Recognize various restrictions on replacements for a variable in a given relation. This can include appropriate restrictions on input numbers (domain) or output numbers (range) that come as a result of the situation or as a result of the rule itself. **A19.** Solve open sentences (equations and inequalities) involving one or two variables by organizing guess and test procedures through the use of tables or graphs. **A20.** Develop formal procedures for solving open sentences (equations and inequalities) in one variable. Use addition, subtraction, multiplication, or division of like quantities on both sides of the sentence to form equivalent sentences.
9-12	**A21.** Solve open sentences (equations and inequalities) in one variable through the systematic application of properties to obtain equivalent sentences. **A22.** Use the quadratic formula to solve quadratic equations with real roots. ***A23.** Apply the ability to analyze number relationships to problem situations. Given data in tabular or graphic format, the student should be able to determine a rule for the relationship and use that rule to solve a problem. These relationships may involve two operations in a linear relationship (for example, $2x + 3 = y$) or simple quadratic relationships. ***A24.** Develop understanding and skills in the use of trigonometric functions. ***A25.** Understand and use properties of direct and inverse variation to solve real problems. ***A26.** Develop understanding and skills in the meaning and use of sequences and series. ***A27.** Solve systems of equations and inequalities. Apply this skill to the solution of real problems. ***A28.** Solve quadratic equations and inequalities. Apply this skill to the solution of real problems. * *Indicates an objective that is fully developed only in courses for college-intending students.*

Graphs†

K-3	**A29.** Use simple charts for reference, comparisons, and recordkeeping. **A30.** Construct bar graphs, histograms, and pictograms. The data for these visual displays should come from realistic problem situations that occur within the classroom. (See objective S26.) **A31.** Use the correspondence of numbers to points on a number line as a way of locating or identifying objects. (See objective G32.) **A32.** Explore representation of ordered pairs in tabular and graphical forms, especially the concept of using rows and columns to locate points on a rectangular grid.
4-6	**A33.** Explore the relationship between two-dimensional graphs of data and sets of ordered pairs. This includes interpreting and graphing the data when given the ordered pairs as well as generating ordered pairs when presented a table, graph, or formula. **A34.** Associate points on the coordinate plane (first quadrant) with ordered pairs of whole numbers.

† *See also* Visual Exploration of Data *in the Statistics strand, p. 49, and* Coordinate Geometry *in the Geometry strand, p. 40.*

Graphs *(continued)*

7-8	**A35**. Recognize and construct graphs of sentences (equations and inequalities), using a number line for linear sentences in one variable and the coordinate plane for linear sentences in two variables. (See objective G35.)	**A36.** Associate points on the coordinate plane (all four quandrants) with ordered pairs. (See objective G36.) **A37.** Explore the plotting of linear and quadratic functions, step functions, and constant functions arising out of real data.
9-12	***A38.** Identify linear functions from their graphs using slope and intercepts. (See objectives G39, G40 and G41.) **A39.** Plot linear and quadratic functions, step functions, and constant functions.	***A40.** Recognize properties of functions (minimums, maximums, roots, asymptotes, etc.) from their graphs. ***A41.** Relate transformations of functions to shifts in their graphs. * *Indicates an objective that is fully developed only in courses for college-intending students.*

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Discrete Mathematics

7

Discrete Mathematics Overview
Discrete Mathematics Objectives

Discrete Mathematics Overview

Discrete mathematics is not a new branch of mathematics. It has been known to and taught by college and university faculty for a number of years, often under the title of finite mathematics. What is relatively new is the awareness that discrete mathematics has a number of very important applications in today's information and technological age. Many, but not all of the applications are relevant to the study of computer science. Because of its importance, a number of prestigious boards and commissions have recommended its inclusion in school programs. For example, the National Science Board Commission on Precollege Education in Mathematics, Science and Technology states in *Educating Americans for the 21st Century*, "Discrete mathematics, elementary statistics and probability should now be considered fundamental for all high school students."

For the first time a discrete mathematics strand is included in a Wisconsin curriculum guide. Acknowledging its novelty, while at the same time recognizing that in future versions of Wisconsin mathematics curriculum guides the role of discrete mathematics will probably be enhanced, the subject is now included for several reasons. First, there is growing recognition of its importance, with mathematicians increasingly recommending that it be included in school programs. Second, it should be pointed out that some relevant aspects of discrete mathematics are already being taught, even though teachers may not have realized that fact. Third, if one acknowledges its importance and its need to be included within the mathematics curriculum, it is necessary to indicate what objectives may reasonably be attained by learners at all grade levels.

Similarity to Other Strands

In a certain sense, discrete mathematics is not much different from the other strands of this guide. Although the terms algebra, geometry, and statistics are more familiar, one needs to recognize that there are many aspects of those subjects which are far too advanced and technical for elementary and secondary school students and are best left for advanced study in higher education. What has been suggested for those subjects in this guide are simply those objectives which are believed to be accessible to school age students. The same is true of discrete mathematics. The objectives suggested are those concepts and skills that are considered within the scope of most, if not all, students, and which, as pointed out above, may already be taught in schools under the rubric of some other subject title. This latter point highlights the fact that much of discrete mathematics is not isolated unto itself, but is highly interrelated with other aspects of mathematics.

. . . much of discrete mathematics is not isolated unto itself, but is highly interrelated with other aspects of mathematics.

Algorithmic Thinking

Much of discrete mathematics involves algorithmic thinking. Nearly everyone is acquainted with the algorithms for adding, subtracting, multiplying, and dividing whole numbers. Those algorithms are procedures to be followed, step-by-step, to produce a desired result. The two-step procedure for averaging a set of numbers is an example of an algorithm that can be used to solve all problems of a particular class. In another type of algorithm, certain steps are repeated. An algorithm for alphabetizing a list of names is an example. The divide-and-average method for approximating the square root of a number is another example. The algorithm is:

Much of discrete mathematics involves algorithmic thinking.

1. estimate to produce a first approximation of the square root;
2. divide the original number by the approximation to produce a quotient;
3. average the quotient and the approximation to produce a new approximation;
4. return to step 2 or stop.

Steps 2 and 3 can be repeated as many times as necessary to produce the desired degree of accuracy. The output of each repetition becomes the input for the next repetition. This repetition is sometimes called iteration; sometimes recursion. Algorithms in which such repetition occurs are called iterative, or recursive algorithms.

Algorithmic thinking has two aspects. It involves the search for an algorithm to solve a problem or the creation of a new algorithm to solve a problem for which an algorithm is not known. It also involves the analysis of an algorithm to determine whether the result of the algorithm is really the solution to the problem, and whether or not the algorithm can be improved to generate solutions more efficiently.

Discrete Variables and Whole Number Arithmetic

Technically speaking, a discrete variable in mathematics means one that takes on whole number values only. This is in contrast to the continuous variables of algebra, calculus, and analysis that can assume any real number value. While this distinction is no longer as clear-cut in present day interpretations and applications of discrete mathematics, it is still true that much of what is studied, and what can be studied at the school level, deals with whole number arithmetic. Number sequences are of interest, as are number theory topics such as divisibility, greatest common divisor, and least common multiples. A great deal of study and exploration on an intuitive level can be carried out with these ideas at the elementary level before they are studied more formally at the middle school and high school level. At the secondary level, permutations and combinations, the counting principle, and the binomial theorem are examples of some whole number arithmetic and algebra topics that fall within the domain of discrete mathematics.

Logic

. . . the ability to use correct logical reasoning becomes one of the important aspects of the subject.

Because one of the major applications of discrete mathematics is the study of computers (although it is not the major reason for study of discrete mathematics, as some might argue), the ability to use correct logical reasoning becomes one of the important aspects of the subject. The argument that the placement and study of logic belongs solely within its traditional home—high school geometry—is no longer valid. The study of logical principles and some formal symbolic logic should be pursued in its own right, independently of other courses, and then applied to various parts of mathematics at a later time. In the early grades, logic can be studied informally as a part of working with sets—sorting, describing, forming unions and intersections, and so on. Somewhat more formal investigation of terms such as and, or, not, and if . . . then can be carried out in intermediate and middle school years. Use of informal rules of inference can also be studied at this time. In the secondary school, mathematical induction is an especially relevant topic to be studied.

Geometric Aspects

Discrete mathematics can have a strong geometric component to it as well. Network diagrams are an excellent way to represent a variety of relationships that may exist among a number of variables. Simple, or even more complex, analyses of a finite number of choices, as in a game like tic-tac-toe or NIM, can be effected using tree diagrams. Flow charts are geometric ways of representing certain classes of algorithms. Primary age children can use simple examples of these kinds of geometric representations by making maps which represent rooms in a school or houses and streets in a neighborhood.

Calculators and Computers

In classroom practice, there is a great deal that can be done with calculators, particularly at the elementary and middle school levels. When using a calculator to compute, or to use one of its special features such as repeated addition or a constant multiplier or to do multiple operations, the child must learn to communicate with a machine in a very precise fashion, following an exact sequence of steps. This is very much akin to programming. A useful exercise is to write down this sequence of steps, using either symbols or geometric representations of the calculator keys, and then have someone else follow those steps.

The advent of high-speed computing devices, which essentially use simple rules of logic and repeated simple calculations, has forced people to reexamine traditional mathematics. This has helped to create the renewed interest in discrete mathematics. There is, for example, an algorithm (which can be done by hand—it does not require a computer) that finds the greatest common divisor of two unequal numbers by using the sole operation of subtraction. As students use such an algorithm and decide why it works, they use arithmetic, number theory, and logic, and, presumably, in the process learn something about division that they may not have learned otherwise. Discrete mathematics asks students to think of such questions as well as whether this algorithm is more efficient than other known algorithms that also find the greatest common divisor. Discrete mathematics can provide a new perspective to familiar topics that have traditionally appeared elsewhere, as well as tie together different branches of the subject.

Table 7

Discrete Mathematics Objectives

Discrete Mathematics Objectives		
K-3	**D1.** Continue simple sequences, given enough elements to determine the pattern or rule. The sequences may be determined by numerical or geometric properties or by other attributes such as color or orientation. **D2.** Sort sets on the basis of one attribute.	**D3.** Follow simple directions written in ordinary English and involving at most four steps. **D4.** Carry out a series of computations using a calculator, involving addition and/or subtraction and involving up to four steps.
4-6	**D5.** Sort sets on the basis of two or three attributes. Determine the union and intersection of two sets. Describe the union by proper use of the logical connective "or" and describe the intersection by proper use of the logical connective "and." **D6.** Determine the outcome of a simple flow chart involving at most two decision statements and at most one nested loop.	**D7.** List the steps for a simple algorithm. **D8.** Carry out a series of computations with a calculator, involving any of the four fundamental arithmetic operations and involving up to six steps.
7-8	**D9.** Use iterative algorithms such as the divide-and-average procedure for finding square root and the Euclidean or subtractive algorithms for determining the greatest common divisor of two numbers.	**D10.** Carry out any series of calculations with a calculator, involving the four fundamental operations, raising to a positive integral power, and extracting roots.
9-12	**D11.** Use a recursive formula to determine the first ten elements of a sequence. **D12.** Represent an algorithm by a flow chart. The flow chart can have up to three decision statements and up to two nested loops. **D13.** Evaluate algebraic expressions involving literal variables and exponents. Evaluate common formulas that are used in geometric measurement, business applications, and science.	**D14.** Write a formula using algebraic symbolism for arithmetic, geometric, and other easily recognizable number sequences. **D15.** Construct and interpret simple tree diagrams. **D16.** Use the rules of logic and symbolic logic to determine the validity of simple arguments. ***D17.** Use mathematical induction to establish truth of selected theorems. * *Indicates an objective that is fully developed only in courses for college-intending students.*

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Problem Solving

8

Problem Solving Overview

Problem Solving Objectives

Problem Solving Overview

In 1980 the National Council of Teachers of Mathematics published a document titled *An Agenda for Action: Recommendations for School Mathematics of the 1980s*. Recommendation 1 of this report states, "Problem solving must be the focus of school mathematics in the 1980s." The National Council of Supervisors of Mathematics, in its 1977 Position Paper on Basic Mathematical Skills, states, "Learning to solve problems is the principal reason for studying mathematics." Similarly, a 1975 Wisconsin Department of Public Instruction publication, *Toward Excellence in Education*, states: "The most important goal of mathematics instruction is the development of student ability to solve problems."

As important as these statements are, they do not reflect a novel position. Problem solving has long been described as a primary goal in mathematics education. Problem solving encompasses many routine and nonroutine processes considered to be essential to everyday living. Therefore, the curriculum must prepare individuals to deal with problems they now face or are likely to face in their future. Also, it must be recognized that what is a problem to one individual may not be a problem to another and that the most appropriate technique for solution may differ from individual to individual. During their mathematics education students should learn to apply the rules of logic necessary for arriving at valid conclusions, and to do so they must be able to seek out and sort relevant facts or factors. The student must be able to define and understand a problem, devise a plan for solution, carry out that plan, and when conclusions are made, subject them to scrutiny.

Problem solving is a process that should permeate all mathematics instruction and student learning.

New Emphasis

The new emphasis being placed on problem solving involves changing the focus from exclusively finding answers to word problems to the acquisition of many problem-solving skills and strategies. Students should then be able to apply these strategies to a wide variety of unfamiliar problem situations. This changing emphasis has brought a need for new teaching materials and techniques. Many of the currently available resources are listed at the end of this section and it should be noted that new, well-prepared materials are regularly appearing.

Study of Problem Solving in its Own Right

This guide recognizes the place of problem solving in the mathematics curriculum in two ways. Problem solving is a process that should permeate all mathematics instruction and student learning. Therefore, problem solving should be evident in all the content strands along with skills and concepts. Secondly, in this section the processes of problem solving are separated from other content sections to highlight them and to point out that the development of problem-solving ability is to be viewed as a legitimate goal in its own right.

Basic Assumptions

The listing of problem-solving objectives is made with the following understandings.

- Most problems can be solved by a variety of strategies.
- No one strategy works for all problems.
- No one approach to a particular problem is the best for all students.
- Problem solving can not be viewed exclusively as a set of distinct skills to be mastered. In fact, the whole of problem solving is much greater than the sum of the parts.

Since a particular problem-solving strategy may not be as useful to one student as it is to another, students will be more powerful problem solvers if they have acquired all the skills and can use the strategies suggested in table 8.

Table 8

Problem Solving Objectives
Grades K-12

*Problem Solving**

P1. Identify when a problem exists and devise problems on one's own.

P2. Read and understand problems. This incorporates the ability to state what is to be found, to state what is given, to identify relationships among components within a problem, to recognize when needed information is missing or when given information is extraneous, and to restate the problem.

P3. Make various representations of a problem. This incorporates the ability to model objects, ideas, or problems with drawings, chips, or tally marks, to make organized and complete lists or tables, and to select and generate mathematical phrases and sentences to represent and solve a problem.

P4. Recognize and use patterns, and make generalizations.

P5. Establish subgoals for solving parts of a problem.

P6. Construct and use simpler versions of a problem. This incorporates the ability to construct and use problems with smaller numbers as a way of identifying necessary operations, to recognize that problems which look different may be solved in similar ways, and to construct related versions of a problem (for example, through limiting or expanding cases which need to be considered).

P7. Use the action present in a problem situation. This incorporates the ability to sequence picture problems, to act out problems, and to conduct experiments or simulations that are appropriate for the solution of the problem.

P8. Obtain data from varied sources such as pictures, tables, and graphs, and relevant resources (other than textbooks). Gather and use samples to obtain data that suggest approximate answers to problems.

P9. Check and revise solutions. This incorporates the ability to check solutions against problem conditions, to guess, check and revise, to check solutions for reasonableness, and to generate candidates for solution and check.

P10. Extend and generalize problem solutions.

* *Students may apply the following objectives to many different types of content. Each objective should be addressed at the appropriate degree of complexity at each grade level.*

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Problem solving activities for students are readily available through Creative Publications, Dale Seymour Publications, and other reputable companies distributing educational materials in mathematics. Consult their latest catalogues.

The Secondary Program

9

Mathematics Graduation Standard
Program Options
Program Implementation
Preparation for Postsecondary Education
Preparation for Calculus

Mathematics Graduation Standard

Sec.118.33, Wisconsin Statutes, as created by Chapter 411, Laws of 1983, establishes statewide graduation standards and minimum credit requirements, including two credits of mathematics for all students graduating after September 1, 1988. The law stipulates that the state superintendent must establish requirements for the courses which are listed in the standards.

Chapter PI 18 of the Wisconsin Administrative code states that in order to graduate students must complete "two credits of mathematics which incorporate instruction in the properties, processes, and symbols of arithmetic and elements of algebra, geometry, and statistics." Due to the inclusion of statistics in the content specifications, virtually all students will be affected, including those of high, average, or low ability.

Elements of algebra, geometry, and statistics are described in general terms in The Informal Mathematics Program, p. 84. More detailed recommendations appear in Appendix K, *A Three-Year Curriculum for Non-College-Bound Students*.

The Wisconsin Mathematics Curriculum Task Force recommends a minimum of one year of algebra, one semester of geometry, and one semester of statistics for all high school students.

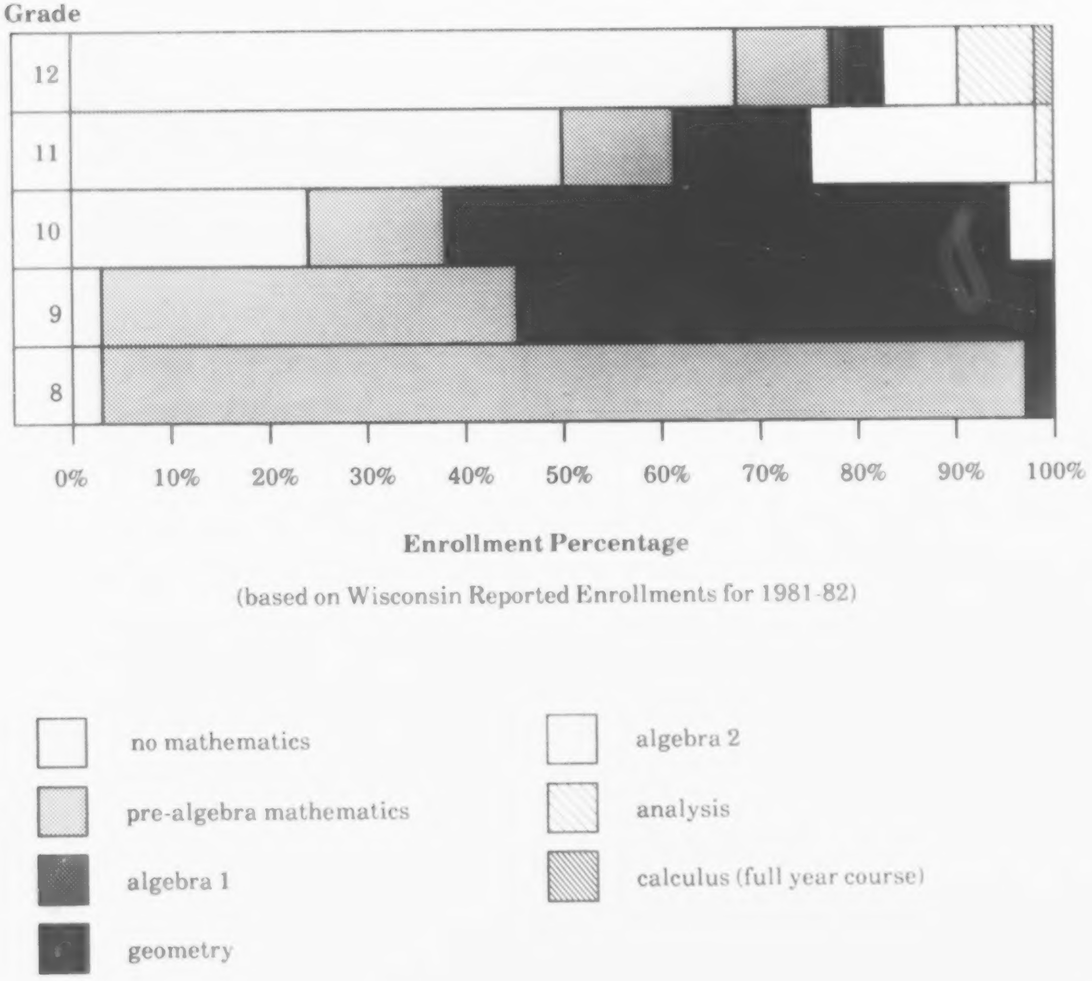
Discrete one-semester courses in each of the four areas are not required, but would meet the requirement if a district determines that such a program is best. The Wisconsin Mathematics Curriculum Task Force recommends a minimum of one year of algebra, one semester of geometry, and one semester of statistics for all high school students. The task force also recommends that mathematics content in high schools be organized into integrated courses—courses with a balanced emphasis on geometry, statistics, algebra, and applications of arithmetic. There are many possible ways to organize instruction to meet these guidelines. Three alternatives are described in the next section.

Program Options

The standard sequence of high school mathematics courses (algebra 1, geometry, algebra 2, analysis) that has emerged over 150 years has successfully prepared many students for the study of calculus. Students who complete this four-year sequence are well prepared to study mathematics successfully in postsecondary institutions. Students who complete three years of this sequence are generally well prepared to succeed in a variety of postsecondary programs. Students who complete two years or less of mathematics are less prepared in mathematics, either because they take too little mathematics or because too much time elapses between their study of mathematics in grades 9 and 10 and their attempt to use mathematics after graduation. Table 9 summarizes enrollment patterns in high schools in Wisconsin.

Table 9 ■

Enrollment Patterns in Wisconsin High Schools



(based on Wisconsin Reported Enrollments for 1981-82)

There are four distinct populations in Wisconsin high schools today. The population of accelerated students takes the college-preparatory sequence ahead of the normal schedule. This population comprises about 5 percent of the students in grade 8 and declines to 1 percent in grade 12. The population taking the college-preparatory sequence on schedule consists of 50 percent of the students in grade 9 and declines to about 10 percent in grade 12. A third population takes the college preparatory sequence one year late. This involves about 20 percent of the students in grade 10 and falls off to about 5 percent in grade 12. The fourth group, about 3 percent in grade 9 increasing to 65 percent in grade 12, takes no mathematics. About 13 percent of eleventh and twelfth graders are enrolled in a pre-algebra mathematics course.

Less than half the students in high school are enrolled in the algebra 1, geometry, algebra 2, analysis sequence.

Less than half the students in high school are enrolled in the algebra 1, geometry, algebra 2, analysis sequence. In this respect this sequence seems appropriate for less than half the students. This is not surprising. The sequence does what it was designed to do. It prepares students for calculus. But only a relatively small number of students take calculus, either in high school or in college. Other students have other needs. For this reason, the role of this traditional sequence in the secondary mathematics program must be reevaluated.

There is another reason to reevaluate the traditional sequence. Even though approximately 45 percent of Wisconsin students are enrolled in that sequence, many of those students are not mastering the content of those courses. Table 10 shows data from the National Assessment of Educational Progress on the achievement level of students who have completed one and two years of algebra.

Table 10

Performance on Selected Algebra Skills

Item	1 year*	2 year*
Solve for x: $3x + 6 - 14 = x + 2$	44%	63%
Factor: $x^2 - 5x + 6$	28%	43%
$3/2a + 3c/4b =$	17%	30%

* *years of algebra completed*

Three-fourths of Wisconsin's students enroll in Algebra I, but less than half of those who enroll are successfully learning the fundamental skills and concepts taught in that course.

In geometry the results are similar. Only 19 percent of seventeen-year-olds were able to determine the area of a parallelogram when given the lengths of its sides and its altitude. Only 43 percent were able to determine the measure of an angle of a triangle when the measures of the other two angles were given. Teaching of proof has been successful with only about half of the 50 percent of students who enroll in geometry. According to Senk, "At the end of a full-year course in geometry in which proof writing is studied, about 25 percent of the students have virtually no competence in writing proofs; another 25 percent can do only trivial proofs; about 20 percent can do some proofs of greater complexity; and only 30 percent master proofs similar to the theorems and exercises in standard textbooks." (Senk, 1985)

These results on algebra and geometry items suggest that many students who now enroll in these courses could be better served by courses in which the focus is either on other mathematics or on other approaches to the same mathematics; less formal approaches.

The large number of students who do not now enroll in the precalculus sequence and the large number who do enroll but who do not master the basic concepts provide a compelling rationale for a restructuring of the secondary school curriculum. Some alternative models have been proposed.

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The University of Chicago School Mathematics Project

The University of Chicago School Mathematics Project (UCSMP) is developing a secondary school curriculum according to the model shown in table 11.

Table 11

UCSMP Secondary School Curriculum Model

Grade	Population 1	Population 2	Population 3*
12		trigonometry, elementary functions, discrete mathematics, analytic geometry	precalculus math
11	1/2 computers 1/2 statistics	world algebra II with computers	algebra II with trigonometry
10	1/2 applied algebra 1/2 applied geometry	world geometry	geometry
9	consumer mathematics	1/2 computers 1/2 statistics	1/2 computers 1/2 statistics
8	grade 8 with calculators	world algebra I	algebra I
7	grade 7 with calculators	1/3 arithmetic 1/3 pre-algebra 1/3 pre-geometry	1/3 arithmetic 1/3 pre-algebra 1/3 pre-geometry

**Advanced placement students take population 3 curriculum one year earlier.*

In this model, population 3 is that group of students who succeed in the present precalculus program. They are characterized as college-bound into natural science or other majors requiring mathematics beyond calculus, or college-bound into institutions with highly competitive admissions. Population 2 students are college-bound into business, social science, or other majors in which calculus does not play as prominent a role, or students who are college hopeful. Population 1 students are not college-bound. About 10 percent of students are not classified due to special circumstances, for example, learning disabilities.

According to Zalman Usiskin (1984), World Algebra I is derived from Algebra I by deleting contrived word problems, trinomial factoring, and manipulations of complicated polynomial and fractional expressions. World Algebra I also includes more applications and an extensive use of calculators. World Geometry deletes a rigorous approach from first postulates, anything more than six weeks devoted to proof, and advanced theorems on circles and triangles. It incorporates transformations for congruence and similarity, symmetry, more applications, more use of coordinates, and more three-dimensional geometry. World Algebra II excludes complicated manipulations done by hand, log tables, intricate graphing, and permutations and combinations. It includes more applications, matrices, general theorems regarding graphing, and use of computers for complicated manipulations and graphing.

UCSMP is potentially flawed by a structure which places low achieving students in one program, and all other students in other programs. This results in de facto homogeneous grouping which research studies suggest is detrimental to the achievement of low-ability students (Petersen, 1984). In addition, any program requiring three different courses at a grade level is difficult to implement in small schools (of which Wisconsin has many).

The California Plan

In addition to a program of mathematics offering preparation for calculus, the *Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve* outlines a three-year program which "requires students to learn mathematics beyond the kindergarten through eighth grade program . . . without requiring the student to begin a traditional college preparatory sequence. These courses would cover much of the content of Algebra 1 and Geometry . . . ." The courses emphasize applications and use an instructional approach appropriate for students with a concrete rather than abstract learning style.

Each course integrates content from measurement, geometry, statistics and probability, logic, and algebra. The third course includes consumer/business applications, but its primary goal is preparing young adults for living in a technological society that requires a high level of mathematical understanding.

This recommendation, coming from a large state in which many textbooks are sold, virtually assures the availability of integrated textbooks for average and below-average students in grades nine through twelve.

The California Plan appears to avoid the homogeneous grouping of low-ability students by providing two programs, rather than the three provided by the University of Chicago Plan. However, if in its implementation most average and above average students are scheduled into the college preparatory sequence, de facto homogeneous grouping will again result.

The Wisconsin Plan

In The Wisconsin Plan students are heterogeneously grouped into classes. Individual differences are addressed within the context of teacher-centered whole group instruction, and enrichment rather than acceleration is the preferred manner for accommodating gifted students. Where the size of the school permits, two different programs are available to accommodate different student needs and interests. Rather than be assigned to programs on the basis of their ability, students select the program that best suits their needs and interests.

The informal curriculum addresses the mathematical needs of most adults. It is the curriculum that most students will elect. For students anticipating careers in mathematics related fields, a more formal curriculum is available that is similar, but not identical to current college preparatory programs.. The curricula are designed with the expectation that students will study mathematics for at least three years (in grades 9-12) no matter which program they select.

An integrated structure, in which some arithmetic, some statistics, some geometry, and some algebra is included each year, is recommended (see table 12). While the structure of the two programs is the same, the content and the level of abstraction are different.

An integrated structure, in which some arithmetic, some statistics, some geometry, and some algebra is included each year, is recommended.

Table 12

The Wisconsin Plan

Grade	Informal Program				Formal Program			
12 11 10 9 8 7	arithemetic	algebra	geometry	statistics	arithemetic	algebra	geometry	statistics

This arrangement capitalizes on the natural interrelationships among the various subdisciplines of mathematics and also allows the easier topics from each subdiscipline to be taught prior to the harder topics from other subdisciplines.

Districts committed to a more traditional course structure can incorporate most of the features of The Wisconsin Plan by using the Modified Wisconsin Plan (see table 13).

Table 13

Modified Wisconsin Plan

Grade	Informal Program		Formal Program
12	elective		precalculus mathematics
11	geometry	algebra	algebra with statistics
10	geometry	algebra	geometry with statistics
9	statistics	algebra	algebra with statistics

Students in either the informal program or the formal program will meet the state graduation requirement, and students in either program could meet entrance requirements to most colleges and universities.

Like the UCSMP plan, students now successful in the precalculus program (perhaps 15 to 30 percent of the students on a statewide average) would continue in a similar program. The precalculus program, the formal program, is the program for mathematically talented students. The informal mathematics program is the program for the majority of students.

The Informal Mathematics Program

The informal mathematics program stresses the utility of mathematics in daily living and in careers which do not require advanced mathematics. The general goals of the program in the areas of arithmetic, algebra, geometry, and statistics are as follows.

Arithmetic Goals

Students should be able to:

- recognize many different uses of numbers;
- associate arithmetic operations with actions or situations;
- express numbers in equivalent and approximate forms;
- use approximation to estimate the results of computation with whole numbers, fractions, and decimals;
- use a calculator to compute with whole numbers, integers, and decimals and use mental arithmetic or pencil and paper algorithms where appropriate;
- solve routine problems including one-step problems, multiple-step problems, rate problems, and percent problems.

Algebra Goals

Students should be able to:

- use variables and the language of algebra;
- find an unknown value using a formula;
- simplify elementary algebraic expressions;
- solve linear equations in one variable;
- associate equations with their graphs;
- represent problem situations by open sentences.

Geometry Goals

Students should be able to (in both two and three dimensions):

- recognize and name common geometric figures;
- identify properties of common geometric figures;
- determine measurement properties of common geometric figures;
- recognize relations among common geometric figures;
- apply properties of figures and relations among figures to determine unknown measures or properties.

Statistics Goals

Students should be able to:

- collect and organize data into tables, charts, and graphs;
- read tables, charts, and graphs;
- use central tendency measures and dispersion measures;
- understand simple probability statements;
- use statistics in decision making.

Each of the goals is addressed during each year of instruction. This program has much of the mathematics content of the California Plan (described in greater detail in the *Mathematics Framework for California Public Schools Kindergarten Through Grade Twelve*) and much of the content of the University of Chicago School Mathematics Project for population 1 and some of the content of UCSMP for population 2. For a fuller description, see Appendix K, *A Three-Year Program for Non-College-Bound Students*.

Based on the enrollment and achievement statistics reported earlier, a school district with a *normal* student population might expect 60 percent to 70 percent of those students to enroll in the informal program. This distribution would have several advantages.

Many average-ability students are not successful with abstract mathematics in competition with higher-ability students. They will have a higher probability of success in classes where a more concrete approach to mathematics is used.

Second, low-achieving students can benefit from being scheduled into classes with average-ability students (see Homogeneous/Heterogeneous Grouping, p. 131).

Third, classes in the formal program can move at a faster pace and cover more topics more thoroughly if average-ability students are in the informal program.

Average-ability students should be allowed to elect to enroll in the formal program, but they should be advised that the courses are demanding and that course standards will be upheld.

Under the Wisconsin Plan (or the Modified Wisconsin Plan), students who successfully complete three years in the informal program may enroll in one of the courses of the formal program or in an elective created as a fourth year course. They may, at this point, elect to discontinue their study of mathematics. Students who have done well in the third year of the informal program may indeed be prepared to enroll in the third year of the formal program. They may be better prepared than many students who now enroll in Algebra 2 (see table 10, p. 80).

Other students in the informal program may successfully enroll in the first or second course in the formal program in their third or fourth year. Many options for crossing from one program to the other exist at each grade level.

The Formal Mathematics Program

The formal mathematics program stresses mathematical preparation for technical post-secondary education in vocational/technical schools, colleges, and universities. The program includes much of the preparation currently offered in algebra 1, algebra 2, and geometry and also includes a substantial statistics component. For many students this program includes a pre-calculus course and for some students calculus may be identified as the appropriate senior year elective. By beginning the program in grade eight, integrating the content, increasing the pace of instruction, and incorporating calculators and computers where appropriate, it should be possible to include one semester of statistics along with the topics now taught without increasing the time for instruction. For a fuller description of the statistics component see table 4, p. 48, and table 5, p. 51.

Many reports have described the high school mathematics needed for postsecondary education. The objectives identified in sections 2 through 8 do not include all those recommended. Planners of curriculum in advanced mathematics should consult the recommendations of the College Board (1983, 1985); the Mathematical Association of America (Appendix I); the Joint Council on College Preparation (1983); the University of Wisconsin System Mathematics Placement Test (Appendix J); and the Office of Undergraduate Orientation, University of Wisconsin-Madison. A detailed description of a program to prepare students for calculus follows on p. 90.

Electives

The number and nature of senior year electives will depend on the size of the school and the interests of the students. In medium to large schools it would not be unusual to find electives in analysis and calculus for students in the formal program. A course in discrete mathematics might be offered as an alternative. Depending on how the program for grades seven through eleven has been structured, it might be appropriate to provide a semester course in statistics (with or without probability). Other possible electives include mathematical problem solving, matrix algebra, and logic.

Students in the informal program who make a belated decision to prepare for a technical career or a college which requires a year of algebra and a year of geometry for admission may want to elect a course designed especially for this purpose or a course from the formal mathematics sequence. Some students planning to take tests for education or employment may elect courses which address the skills represented on those tests.

Program Implementation

The central feature of the Wisconsin Plan is not its curriculum for college-bound students nor its organization of curriculum into integrated courses. The recommendations for college-bound students are very similar to programs now in place. The organization of curriculum into integrated courses is a preferred option, but not an essential feature of the plan. The central feature is the provision of a strong program of instruction in algebra, geometry, and statistics for non-college-bound students.

The impoverished curriculum now available for non-college-bound students is not a well-kept secret. The Conference Board of the Mathematical Sciences (1982) has observed that:

The impoverished curriculum now available for non-college-bound students is not a well-kept secret.

> For the many students in secondary school who are not specially talented in mathematics and not headed for careers in science or technology, current programs are a source of discouragement, anxiety and repetition in a dull "basic skills" program which serves them poorly. We cannot ignore the needs of this large and important group.

The National Council of Teachers of Mathematics, in its *An Agenda For Action: Recommendations for School Mathematics of the 1980s* (1980), also commented on this weakness of school mathematics programs:

> In many current programs, the student who does poorly in the algorithmic skills finds progress in all aspects of mathematical development halted, since remediation is designed to concentrate solely on this deficiency.

Participants at the Kenwood Conference (Chambers, 1983) expressed the attitude of the Mathematics Curriculum Task Force and the central philosophy of this curriculum guide when they listed among their Assumptions of the New General Mathematics:

> Children can and will learn mathematics if provided an appropriate program.

Having made a commitment to develop a stronger program in mathematics for non-college-bound students, a district may elect to include or reject other features of the Wisconsin Plan or to incorporate one or more of those features over a period of years. An integrated curriculum is desirable pedagogically, but difficult to implement immediately because of the shortage of texts which can be used as instructional materials. This forces the use of a multi-text approach and requires the teacher to integrate the separate materials—a difficult job.

Children can and will learn mathematics if provided an appropriate program.

Heterogeneous grouping of students into classes is desirable because it reduces the tendency to provide low-ability students with impoverished curriculum and impoverished instruction. However, many teachers and districts are committed to a policy of homogeneous grouping of students. A preference for homogeneous grouping does not preclude adopting the curriculum advocated for low-ability students. Teacher-centered whole-group instruction is the dominant form of instruction in Wisconsin high schools. Teachers preferring small group or individualized instruction and who are skilled at it should not be discouraged from continuing that practice in order to implement the Wisconsin Plan.

Small districts will have problems of implementation not present in large districts. Wisconsin has 383 school districts with high schools. Of those, 62 percent have a 9-12 enrollment of less than 500 (an average of less than 125 students per grade) and 36 percent have a 9-12 enrollment of less than 300 (an average of less than 75 students per grade). Clearly these districts cannot provide five or six sections of ninth grade mathematics. How can they provide the programs advocated?

A school district which has 60 to 90 students per grade can provide three sections of mathematics at that grade level. If the distribution of student ability in that district is average, we might expect to find 30 percent to 40 percent of the students are college-intending. Many of those students may be interested in the formal mathematics program. One section of formal mathematics and two sections of informal mathematics could be scheduled for each of the ninth and tenth grades. While there may be some shifting of students from the formal mathematics program to the informal program, and also from the informal to the formal, student enrollments should sustain that pattern of scheduling through grade 10. Due to the two-year graduation requirement, three years in some districts, most students will enroll in mathematics in the ninth and tenth grades.

By grade eleven, however, the district can expect considerable attrition of student enrollments in mathematics and may only provide one section of informal mathematics and one section of formal mathematics at that level. At the twelfth grade level the district may assign interested students from the informal program to one of the previous levels of the formal program. An average of 10 percent to 15 percent of Wisconsin students enroll in a fourth-year mathematics course; the district should provide a program for them.

In smaller districts, districts with enrollments of less than 60 students per grade level (26 percent of Wisconsin districts fall into this category), the problem is more difficult. These districts can provide only one or two mathematics sections per grade level. Districts with only one section per grade level should probably offer only the informal program, with individualized enrichment available for college-intending students.

Districts with two sections per grade level should determine what portion of their population, typically, is college-intending. If that figure approaches or exceeds 50 percent the district may want to schedule one section of informal mathematics and one section of formal mathematics in each of grades nine and ten. Otherwise the district has two options: it can schedule two sections of informal mathematics and provide individualized enrichment for the college-intending students, or it can decide to fund a third section for those students. The district should not form one section of informal mathematics and one section of formal mathematics and force students into the formal mathematics section to balance enrollments.

Rather than thinking of the non-college-intending students as abnormal and requiring special programming, they are considered the normal students taking the normal program . . . .

Note the attitude reflected in the suggestions for program implementation. Rather than thinking of the non-college-intending students as abnormal and requiring special programming, they are considered the normal students taking the normal program, the informal program. After all, they typically constitute 60 percent to 70 percent of the student population. It is the college preparatory students in most districts who require special programming, and that programming should be provided whenever possible.

Preparation for Postsecondary Education

Students who complete the program recommended for graduation from high school (see Appendix K, *A Three-Year Curriculum For Non-College-Bound Students*) will have a good foundation in mathematics and will meet the mathematics entrance requirement of many Wisconsin colleges, universities, and technical schools. They will not be able to select undergraduate majors with mathematics prerequisites, however, without first completing one or more additional mathematics courses. For this reason, students planning to attend college should be aware that the formal mathematics program offers more options for undergraduate study and for careers. Similarly, students preparing for technical careers in technical colleges should consider the formal mathematics program.

Students interested in applying to selective colleges and universities are advised to have at least three years of formal mathematics.

Students interested in applying to selective colleges and universities are advised to have at least three years of formal mathematics. Many institutions require one year of algebra and one year of geometry for admission. This requirement is less rigid than it once was, however, due to the trend toward integrated mathematics courses in high school.

Much of the literature on preparation for college is devoted to preparation for calculus since calculus is frequently identified as the entry-level mathematics course. Since many college students do not study calculus, the recommendations of this guide do not assume that preparation for college and preparation for calculus are identical. Resources on mathematics preparation for college have already been mentioned (see p. 86) and students and advisors, as well as curriculum planners are encouraged to familiarize themselves with those reports.

Program planners should keep two additional points in mind. First, the role of calculus in the undergraduate mathematics program is being reviewed. Emphasis is beginning to shift from the continuous techniques of calculus toward discrete methods and the use of computers. During this transitional period, program planners are encouraged to be flexible and to offer a balanced emphasis.

Second, many high school students will not study calculus either in high school or in college. These students also deserve good mathematics programs in high school. Historically, preparation for calculus defined the total high school mathematics program. High schools now have an obligation to provide the best possible mathematics instruction to students of all interests and abilities.

Preparation for Calculus

Many college and university mathematics programs either begin with calculus or with courses designed to prepare students for the subsequent study of calculus. Hence its influence on the content of high school mathematics programs is considerable.

Districts in which a sufficiently large number of students (ten or more, perhaps) may subsequently study calculus are encouraged to offer a program which provides this preparation.

The objectives below are from *Precalculus Mathematics* by John G. Harvey, Professor of Mathematics, University of Wisconsin-Madison. A revision of that article (Harvey, 1978) was subsequently published in the Mathematics Teacher and reprinted in *The Mathematics Curriculum: 9-12* (Chambers, 1980). Teachers of courses designed to prepare students for the study of calculus are encouraged to read that article.

Students planning to enroll in calculus as their first collegiate mathematics course should attain the performance objectives listed below in the postgeometry, precalculus secondary school mathematics curriculum. The stated performance objectives are terminal ones. No attempt is made to describe all the objectives that must be attained between the geometry course and these objectives, because they probably number in the hundreds if not the thousands.

Polynomial Functions with Real Coefficients

1. Given a set of functions, the student chooses the polynomial functions from that set.
2. Given a polynomial function, the student identifies the degree of the polynomial.
3. Given a polynomial function, the student identifies the value of the leading coefficient.
4. Given a polynomial function of degree n, the student identifies the coefficient of the ith term of that polynomial ($0 \leq i \leq n$).
5. Given a polynomial function and the value of one of the coefficients, the student identifies the term(s) of the polynomial with that coefficient.
6. Given a set of real numbers $a_n, a_{n-1}, a_{n-2}, \cdots, a_1, a_0$, the student writes the polynomial function having a_i as the coefficient of the ith term of the polynomial.
7. Given a positive integer n and a real number a, the student uses the binomial theorem to write the product $(x - a)^n$ in unfactored form.
8. Given the polynomial function and a real or complex number, the student identifies whether that number is a zero of the polynomial.
9. Given a polynomial function p and a real number a, the student uses the remainder theorem to state that the remainder on dividing the polynomial by $(x - a)$ is $p(a)$.
10. Given a quadratic polynomial function, the student completes the square to find the zeroes of the polynomial.
11. Given a quadratic polynomial function, the student uses the quadratic formula to find the zeroes of the polynomial.
12. Given a polynomial function, the student locates the real zeroes of the function between consecutive integer values.
13. Given a polynomial function, the student locates all of the zeroes of the polynomial that are rational numbers.
14. Given a polynomial function, integer bounds on the real zeroes of that function and a positive real number ε, the student approximates each of the real zeroes of the polynomial to within ε.
15. Given a polynomial function and all of the zeroes of that function, the student writes the polynomial in factored form.
16. Given a polynomial function, the student writes the polynomial as a product of linear and quadratic factors so that each factor has real coefficients and so that the quadratic factors have no real zeroes.

Rational Functions

1. Given a set of functions, the student chooses the rational functions from the set.
2. The student constructs examples of rational functions.
3. The student defines a rational function as the quotient of two polynomial functions.
4. Given a rational function $r = p/q$ such that $\deg(p) > \deg(q)$, the student constructs polynomials s and t such that $r = s + t/q$ and $\deg(t) > \deg(q)$.
5. Given a rational function p/q such that $\deg(p) < \deg(q)$, the student writes this function as a sum of partial fractions.

Trigonometric and Inverse Trigonometric Functions

1. Given a set of functions, the student chooses the trigonometric functions from the set.
2. Given a point in the real plane, the student constructs the sine (cosine, tangent, cotangent, secant, cosecant) of the angle determined by the point, the origin, and the positive x-axis.
3. Given a trigonometric function and a real number, the student finds the value of the function at that real number.
4. Given a trigonometric function, the student identifies the period of that function.
5. Given a trigonometric function, the student identifies the amplitude of that function.
6. Given a linear function $ax + b$ and a trigonometric function f, the student identifies the period of the composed function $f \circ (ax + b)$.
7. Given a linear function, $ax + b$ and a trigonometric function f, the student identifies the amplitude of the composed function $f \circ (ax + b)$.
8. The student states each of the following basic identities:
    a. $\sin x = \dfrac{1}{\csc x}$ and conversely
    b. $\cos x = \dfrac{1}{\sec x}$ and conversely
    c. $\tan x = \dfrac{1}{\operatorname{ctn} x}$ and conversely
    d. $\sin^2 x + \cos^2 x = 1$

e. $\tan^2 x + 1 = \sec^2 x$

f. $1 + \operatorname{ctn}^2 x = \csc^2 x$

g. $\sin(a + b) = (\sin a)(\cos b) + (\cos a)(\sin b)$

h. $\cos(a + b) = (\cos a)(\cos b) - (\sin a)(\sin b)$

i. $\sin(-a) = -\sin a$

j. $\cos(-a) = \cos a$

9. Given a trigonometric function and the value of that function at some real number, the student constructs the values of the other trigonometric functions at that real number.

10. Given a real number and a trigonometric function, the student constructs a real number such that the value of the trigonometric function at the constructed number is the given number, for example, given x and sine, construct y such that $\sin y = x$.

11. Given a trigonometric function, the student defines the inverse of the function.

Logarithmic and Exponential Functions

1. Given an even positive integer n, and a nonnegative real number x, the student defines $x^{1/n}$ as the set of real numbers y such that $y^n = x$.

2. Given an odd positive integer n and a real number x, the student defines $x^{1/n}$ as the real number y such that $y^n = x$.

3. Given a positive rational number m/n and a real number x, the student defines $x^{m/n}$ as $(x^{1/n})^m$ and as $(x^m)^{1/n}$.

4. Given a positive rational number m/n and an appropriate real number x, the student defines $x^{-m/n}$ as $1/x^{m/n}$.

5. Given a rational number m/n, the student finds values of the function $x^{m/n}$.

6. Given rational numbers a and b, the student states that

   a. $x^a \cdot x^b = x^{a+b}$

   b. $(x^a)^b = x^{ab}$

7. Given a real number a, the student finds values of the function a^x for rational numbers x.

8. Given a positive rational number a and a positive real number x, the student defines $\log_a x$ as the real number y such that $a^y = x$.

9. Given a positive rational number a and positive real numbers x and y, the student states that

   a. $\log_a(x \cdot y) = \log_a x + \log_a y$

   b. $\log_a(x/y) = \log_a x - \log_a y$

   c. $\log_a(x^y) = y(\log_a x)$

This topic is a difficult one since the definitions of both the exponential and logarithmic functions depend on the concept of continuity and the operation of integration. Thus, at this point in the development of these concepts, a circular definition is necessary, and the functions cannot be defined for all appropriate real numbers.

Analytic Geometry and Linear Algebra

1. Given a real number and a number line, the student identifies (approximately) the point on the line that corresponds to the number.
2. Given a point on a number line, the student identifies (approximately) the real number that corresponds to the point.
3. Given an ordered pair of real numbers and a two-dimensional rectangular coordinate system, the student identifies the point in the coordinate system that corresponds to the ordered pair.
4. Given a point in a two-dimensional rectangular coordinate system, the student identifies the ordered pair of real numbers that corresponds to the point.
5. Given an ordered triple of real numbers and a three-dimensional rectangular coordinate system, the student identifies the point in the coordinate system that corresponds to the ordered triple.
6. Given a point in a three-dimensional rectangular coordinate system, the student identifies the ordered triple of real numbers that corresponds to the point.
7. Given two points on a number line, the student finds the distance between them by taking the absolute value of the difference of the real numbers that correspond to the point.
8. Given two points in a two-dimensional rectangular coordinate system, the student finds the distance between the two points by applying the distance function to the ordered pairs of real numbers that correspond to the points.
9. Given two points in a three-dimensional rectangular coordinate system, the student finds the distance between the points by applying the distance function to the ordered triples of real numbers that correspond to the points.

10. Given a linear function, the student constructs the graph of the function.

11. Given the linear function $f(x) = ax + b$, the student states that a is the slope of the line represented by f.

12. Given the linear function $f(x) = ax + b$, the student states that the y-intercept of the line represented by f is b.

13. Given the linear function $f(x) = ax + b$ and given that $a \neq 0$, the student states that the x intercept of the line represented by f is $-b/a$.

14. Given two points in a plane represented by the ordered pairs (x_1, y_1) and (x_2, y_2), respectively, the student calculates the slope of the line that passes through the points as

$$m = \frac{y_1 - y_2}{x_1 - x_2} .$$

15. Given two points in a plane represented by the ordered pairs (x_1, y_1) and (x_2, y_2), respectively, the student calculates the slope of the line between the two points, the student states that the equation of the line passing through the two points is

$$y - y_1 = m(x - x_1)$$

or

$$y - y_2 = m(x - x_2).$$

16. Given the equations of two lines, the student states whether the lines are parallel.

17. Given the equation of a line having slope m ($m \neq 0$), the student states that the line perpendicular to the given line has slope $-1/m$.

18. Given the equation of a line and a point on that line, the student constructs the equation of the line perpendicular to the given one that passes through the given point.

19. Given the equation of a line and a point not on that line, the student constructs the equation of the line through the given point parallel to the given line.

20. Given a system of m linear equations in n unknowns ($1 \leq m$, $n \leq 5$), the student finds all of the solutions of the system of equations.

In addition to these performance objectives, the student should have experience with ellipses, parabolas, and hyperbolas, with the equations for these three sections, and with parametric descriptions of lines and these conic sections.

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Curriculum Development

10

What Is Curriculum?
The Curriculum Development Process
Professional Standards for Selection and Implementation of Instructional Materials
Textbook Selection

What Is Curriculum?

A school may have more than one mathematics curriculum, if it has a curriculum at all. Commonly, a district makes a textbook adoption and the text then defines the curriculum. If it is expected that all teachers will cover all the material in the text with all students, the textbook becomes the *intended* curriculum—the curriculum the district intends to teach.

Sometimes districts form curriculum committees which develop lists of instructional objectives for mathematics. These lists are distributed among mathematics teachers and are meant to identify the district's curriculum. The lists may or may not vary from the topics in the texts being used. In this case, the list of objectives becomes the intended curriculum.

Individual teachers frequently make unilateral decisions about deviations from the intended curriculum. Therefore, the intended curriculum in most districts is not the curriculum the students are actually taught. The *implemented* curriculum is usually teacher-specific and not uniform throughout the district. This creates problems for both teachers and students as students move from grade to grade.

Perhaps the most accurate measure of a district's curriculum is the mathematics the students actually learn, as measured by their performance on tests. We teach the students many things, but they frequently do not learn all of them. Curriculum thought of in this way is called the *achieved* curriculum.

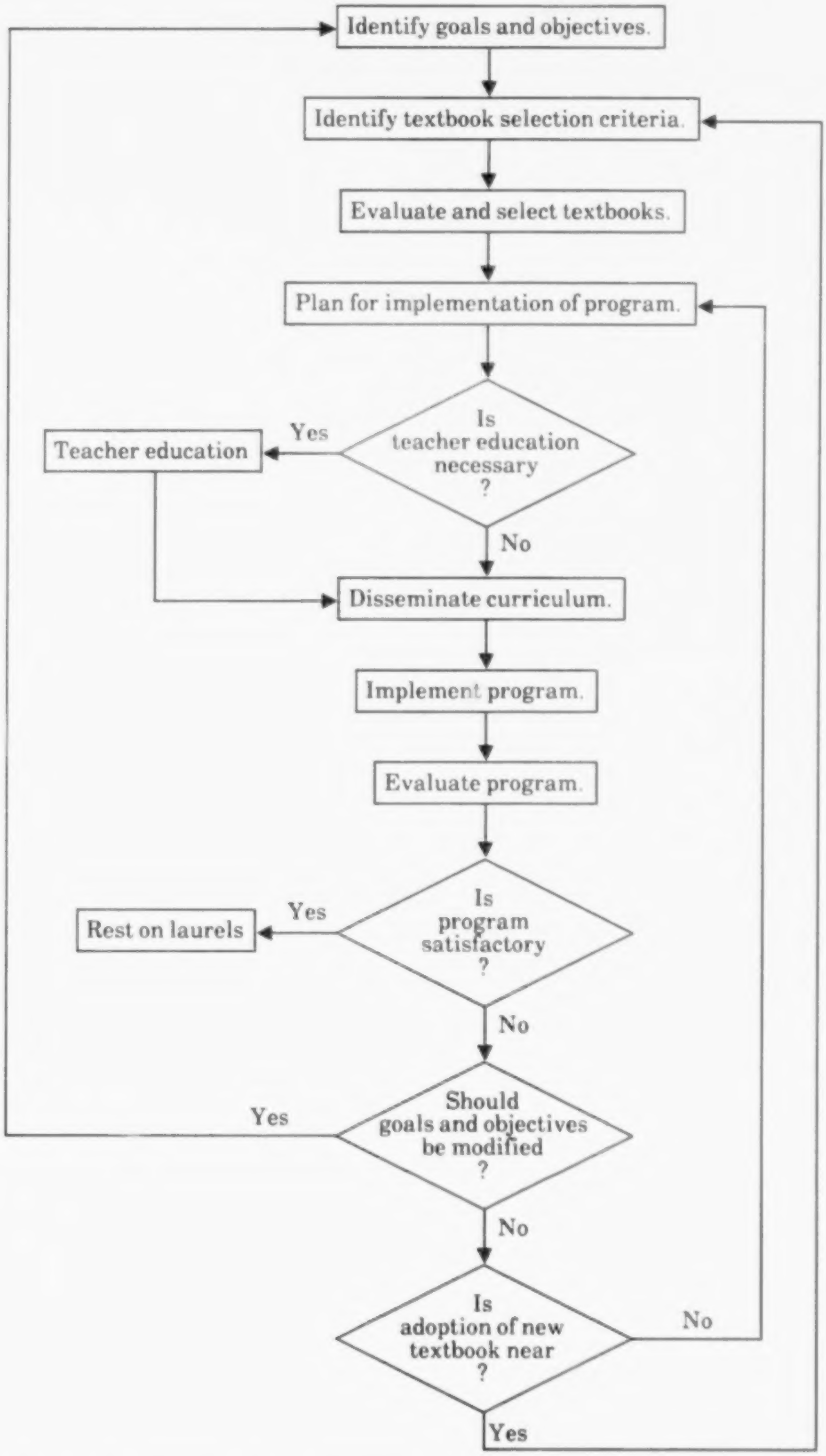
The curriculum development process should be structured in such a way that the intended curriculum, the implemented curriculum, and the achieved curriculum are as nearly identical as possible.

The Curriculum Development Process

General aspects of curriculum development are discussed in *A Guide to Curriculum Development: Purposes and Procedures*, a companion volume to this *Guide to Curriculum Planning in Mathematics*. The curriculum development process described below is generally applicable to mathematics curriculum in grades K-8 and is also applicable to the development of integrated curriculum in grades 9-12.

Curriculum development is a continuous process which proceeds according to the following diagram (figure 3).

Curriculum Development Process


Identify goals and objectives.
Identify textbook selection criteria.
Evaluate and select textbooks.
Plan for implementation of program.
Is teacher education necessary ?
Yes
Teacher education
No
Disseminate curriculum.
Implement program.
Evaluate program.
Is program satisfactory ?
Yes
Rest on laurels
No
Should goals and objectives be modified ?
Yes
No
Is adoption of new textbook near ?
No
Yes

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The process can begin almost anywhere in the cycle. Perhaps the most critical determinants of a starting point are: (1) Have goals and objectives been identified? and (2) Is an adoption of a new textbook near? If we assume, for purposes of this illustration, that the answer to both questions is no, we can start at the top with the identification of program goals and objectives. We then follow the chart until all steps of curriculum development have been completed and we are able to say, "Yes, the program is satisfactory."

. . . the only way out of the infinite curriculum development loop is to achieve satisfactory results on the program evaluation.

Notice that the only way out of the infinite curriculum development loop is to achieve satisfactory results on the program evaluation. Seldom will a district determine that no improvement can be made by modifying one or more program components. The following brief discussion of each component of the curriculum development process may help to clarify each item in the flow chart.

Goals and Objectives

The purpose of sections 1 through 9 is to help mathematics curriculum committees identify their goals and objectives. In rare instances a committee may find these proposed goals and objectives perfectly satisfactory and adopt them intact. More often the committee will want to modify the list by making additions and deletions. References in the appendixes have been provided to support the recommendations of the Mathematics Curriculum Task Force, and those references should be reviewed by the local committee as part of the identification process.

In addition to the recommendations of the task force and the various national committees, local committees may find it useful to have criteria for the inclusion of mathematics topics. Such criteria may be developed by the committee or borrowed from other sources. According to Zalman Usiskin (1980) a topic should be included because of:

- its importance in understanding or coping in society;
- its importance for later work in mathematics; or
- its importance in understanding what mathematics is about.

Textbook Selection Criteria

Without criteria to apply to each text it is easy to become distracted by features which may be irrelevant to the quality of the text as a source of instructional material. A list of suggested criteria appears later in this section. The committee should be sure that this list (or any other proposed list) is first modified to be compatible with the district's goals and objectives. Some characteristics may be determined to be more important than others, and the criteria may be weighted to reflect this. Whatever scale is adopted should be applied to all series being evaluated.

Textbook Selection

It will not be possible to evaluate every textbook series available. Thus some preliminary identification of series to be evaluated is necessary. If the series presently being used has been fairly satisfactory, its contemporary version should probably be considered. Recommendations should be sought from neighboring school districts or from other districts which may be known to have good programs. Series which have collected field data prior to publication or which can provide data from districts using their series may be chosen over those which have no such data. Series favorably reviewed in professional journals such as *The Mathematics Teacher* or *The Arithmetic Teacher* should also be included. It is also a good idea to include at least one series which was developed by a nonprofit agency. Two such series at the K through 6 level are *Developing Mathematical Processes* (Delta) and *Comprehensive School Mathematics Program* (CEMREL, now distributed by McREL).

At least five series should be examined very carefully, and at least six months should be allowed for the review of the series under consideration.

Planning for Implementation

The work of the mathematics curriculum committee frequently stops with the selection of a textbook. This is unfortunate because a textbook is not a curriculum. Without direction teachers do not use their textbooks uniformly and do not work systematically to achieve district goals and objectives. Instead they make unilateral decisions which enable them to achieve their personal goals, which are not necessarily compatible with the district goals and objectives.

. . . a textbook is not a curriculum.

The mathematics committee should study the adopted series to develop a strategy for using the texts to achieve the district goals. The committee applies the district's priorities to the text selected and identifies the core curriculum topics from among all the topics in the text. This is critical because any text worth selecting will contain more material than can be covered in most classrooms within a year. Hence some selection of topics must take place at each grade level. The committee can determine priorities either chapter by chapter or topic by topic within chapters. The latter is preferred because it results in a more balanced distribution of topics over the year and is more likely to meet the recommendations of the Mathematics Curriculum Task Force and the major professional organizations.

If the committee has identified priorities within strands, and it is recommended that it does so, then it is possible to work strand by strand through the texts at all grade levels to identify the relevant high-priority portions of the text at each grade level. Those portions should be identified by page numbers and assigned a recommended number of instructional days. Working by strands across grade levels allows the committee to be sure that appropriate prerequisite activities within the strand are developed at each grade level.

Once the appropriate pages have been identified for the high-priority topics within all strands, it will be necessary for the committee to review the program at each grade level to determine whether the total number of recommended instructional days is less than the total number of days available for instruction. If not, a revision of the topics identified or the recommended time allocations will be necessary so that the recommendations can actually be implemented.

The curriculum document . . . should provide each teacher with the information he or she needs to implement the district's curriculum.

The curriculum document, when completed, should provide each teacher with the information he or she needs to implement the district's curriculum. This would include, as a minimum:

- the identification of those pages of the selected text which contain the core topics, along with a statement clarifying which core topic is being addressed;
- information on the students' previous exposure to each core topic, including an indication as to whether this treatment is introductory (first exposure), developmental (students have been introduced to the topic, but that earlier exposure is being extended to cover new related material), mastery (no further developmental work is planned), or review (students were expected to achieve mastery at an earlier time);
- a schedule for the average number of days needed to cover the topic (and still have time to cover all other core topics);
- minimal expectations of students upon completing this topic;
- suggested evaluation instruments to determine whether the appropriate level of student achievement has been attained;
- an identification of appropriate supplementary materials, along with all information listed above;
- recommendations for teaching the non-core topics to the entire class (if it appears there will be instructional days available after completing the core curriculum) or for use as enrichment with individual students;
- policies related to the content or implementation of the mathematics curriculum, the use of calculators or the assigning of homework for example.

Teacher Education

The implementation of a new curriculum frequently requires more than the distribution of the new curriculum guide. Successful implementation will require that all teachers are confident of their own understanding of the mathematics, that they know how to teach it, and that they are aware of other instructional materials (in addition to the textbook) which are available.

Most teachers at all grade levels K-12 could benefit from some additional study of statistics concepts and methods of teaching statistics through an activity approach. Similarly, if problem-solving goals are to be achieved many teachers will need an opportunity to learn methods of teaching problem solving. It is not unusual for teachers in the elementary grades to feel uncomfortable with geometry topics in the curriculum due to their own limited understanding of geometry or their belief that geometry must be taught in a very formal and abstract manner, as it frequently is in the high school course.

An assessment of the staff's preparation to teach the new curriculum should be used to plan appropriate teacher education activities, and appropriate teacher education should precede implementation. Otherwise implementation districtwide will almost certainly fail. Federal funds are now available to all districts for the professional development of mathematics and science teachers. These funds are administered by the Department of Public Instruction. These funds should be used to assist teachers to implement the local curriculum, and should be augmented by funds from the district.

Dissemination

Dissemination of a new curriculum requires more than the distribution of the new guide and the new books. It requires a fairly thorough discussion of all aspects of the guide. In addition to describing what the curriculum is, it is necessary to give considerable attention to why those curriculum topics were chosen. If teacher education has been provided, this rationale could be included as a part of that curriculum development component. Otherwise, it must be done in connection with dissemination.

Each teacher should be familiar with the general framework of the entire K-12 curriculum and the rationale for the curriculum as it has been identified. In addition to the topics of the curriculum, policies such as the use of calculators and the role of homework should be understood and supported by all teachers.

Each teacher should be familiar with the general framework of the entire K-12 curriculum and the rationale for the curriculum as it has been identified.

Teachers should be thoroughly familiar with the curriculum of the grade they teach and acquainted with the curriculum of the grades immediately before and after. This enables them to be aware of what children have studied previously, and what instruction will follow. This should give teachers a better sense of the continuity of the curriculum. It will also make them more aware of prerequisite skills and concepts at each grade level. This knowledge is critical if a spiral curriculum is to be successfully implemented.

Each teacher should be thoroughly familiar with the core topics of his or her grade and be able to plan for a year of instruction that gives adequate attention to the core topics for that grade.

If districtwide tests are administered to students, the nature of those tests should be made clear, including the instructional objectives for which students will be held accountable.

Teacher support is an important factor in curriculum implementation. Support services available should also be described during the curriculum dissemination phase.

Teacher education and dissemination cannot typically be achieved on the day before school begins. These activities should be conducted in the spring in order to allow for the resolution of differences of opinion on choices, to give teachers adequate time to master the content, and to provide time for teachers to plan for the coming year.

Implementation

Administrative support and peer support are very important, especially during the first year of implementation. Administrative support should be provided along with supervision. Supervision is necessary to verify that the intended curriculum is the implemented curriculum. If evaluation indicates that the goals of the program were not achieved, it will be important to know whether the program was implemented but not achieved, or whether it was intended, but not implemented. If one or more teachers are not implementing the curriculum, early identification of any problems can help to prevent small problems from becoming larger problems later on.

While a full scale evaluation will not take place until the end of the first year, teachers and administrators will be making informal judgments about the success of the program throughout the implementation phase, and should feel free to make adjustments which seem warranted.

Program Evaluation

The curriculum development process is a system in which the program evaluation component not only measures the success or failure of the process but also identifies steps to be taken to produce better student achievement in the future. While it is not uncommon for average performance on standardized tests to be used as the fundamental measure of the quality of an instructional program, this measure is very difficult to interpret. It should not be used to evaluate programs. It can be used as part of a comprehensive pupil evaluation plan.

A suggested list of program evaluation criteria follows. That list should be modified to meet local needs and should then be used as part of the curriculum implementation plan. At the end of the first implementation year an evaluation should be conducted to identify problem areas which, if corrected, could produce greater student achievement. Student enrollment and student achievement are two of the fundamental indicators of the success of a curriculum.

If measures of student achievement on important curriculum topics are unsatisfactory, there are several possible explanations.

- The program was not implemented as planned (for example, some teachers did not teach the geometry objectives and students did poorly).
- The time allocated to certain instructional objectives was insufficient.
- The instructional methodology used for certain instructional objectives was inappropriate (for example, statistics was taught too abstractly through the use of memorized formulas rather than more concretely, with emphasis on student projects).
- The expectations were unreasonably high.

If the program was not implemented as planned, steps to improve implementation can be taken. If the program was implemented as planned, expectations can be modified (some mathematics is very difficult, yet worth doing), instructional time can be reallocated, or different instructional methodology can be tried.

If students fail to enroll in elective mathematics courses, the electives may be inappropriate or the students may not understand the importance of further mathematics training for their future.

The program evaluation criteria which follow in table 14 were developed to provide a comprehensive evaluation of teacher preparation, curriculum, instruction, evaluation, and program implementation.

Like the lists of instructional objectives in this guide, this program evaluation list is a suggested plan. Districts may choose to modify the list prior to using it. Another program evaluation plan, *How to Evaluate Your Mathematics Program*, is available from the National Council of Teachers of Mathematics.

Table 14

Program Evaluation Criteria*

Scale: **1** (Inadequate), **2** (Marginal), **3** (Adequate), **4** (Excellent)

Teacher Qualifications

______ a) All teachers of mathematics are properly certified.

______ b) All K-5 mathematics teachers have at least 12 semester hours in mathematics and mathematics education.

______ c) All 6-8 mathematics teachers have at least 21 semester hours in mathematics and mathematics education.

______ d) All 9-12 mathematics teachers have at least 34 credits in mathematics and at least 6 credits in mathematics education.

______ e) Within the last five years all mathematics teachers have completed at least 2 semester credits in mathematics or the equivalent; or at least 2 semester credits in mathematics education or the equivalent.

______ f) Within the last three years all mathematics teachers have spent at least three full days in professional meetings on mathematics or mathematics education.

______ g) All mathematics teachers are members of a professional organization of mathematics teachers.

Curriculum

______ a) The district has a K-12 mathematics curriculum guide with instructional objectives keyed to instructional materials.

______ b) The district guide has been reviewed by a committee within the last 5 years.

______ c) The curriculum is consistent with state and national recommendations on computation, problem solving, geometry, statistics, use of calculators.

______ d) The curriculum shows evidence of continuity from grade to grade.

______ e) The curriculum assumes child development through concrete, semi-concrete, and abstract stages of learning.

______ f) The curriculum is rich enough for a diverse student population with core, enrichment, and review topics identified.

______ g) The curriculum incorporates the spiral development of concepts within grades and across grades.

______ h) The curriculum has been disseminated to teachers, students, and parents.

______ i) Four years of appropriate course work are available for both college-bound and non-college-bound students in grades 9-12.

Instruction

______ a) A validated model of effective mathematics teaching is used by all teachers.

** In some cases a rating will represent an average. In those cases, instances which fall below the average should be noted so they can be improved. A district should seek ratings of "excellent" in each category. A program improvement plan should be developed to increase ratings in subsequent evaluations.*

______ b) All teachers implement the curriculum by teaching all core topics identified for that course or grade level.

______ c) All teachers use principles of distributed practice through reviews, homework assignments, and cumulative tests.

______ d) All teachers provide individual or whole-group remediation as needed.

______ e) All teachers provide enrichment experiences for students as appropriate.

______ f) All teachers use manipulative materials as appropriate to develop mathematics concepts.

______ g) All teachers make problem solving the focus of mathematics instruction.

Evaluation

______ a) All teachers continuously evaluate pupil learning through direct questioning, observing students at work, reviewing homework, and periodic testing.

______ b) All teachers maintain accurate records of students strengths and weaknesses.

______ c) All teachers use results of pupil evaluation to plan for instruction.

______ d) Districtwide objective referenced tests are administered to all pupils on a regular basis.

______ e) Results from districtwide tests are used for program evaluation.

______ f) The mathematics program is evaluated on a regular basis (at least once very five years) by outside persons.

______ g) Enrollments in elective mathematics courses show no differences based on sex, race, or socio-economic status.

______ h) Achievements in mathematics show no differences based on sex, race, or socio-economic status.

Implementation

______ a) At least 50 minutes per day is allocated for mathematics instruction at all grade levels.

______ b) All teachers are acquainted with the district curriculum, philosophy, and policies and support them.

______ c) A standing K-12 mathematics committee continuously plans, monitors, and evaluates the curriculum and its implementation.

______ d) Time and/or money is provided for curriculum development, program evaluation, and purchase of instructional materials.

______ e) Adequate district resources are available for appropriate professional development activities on a regular basis.

______ f) Supervisors observe, assist, and evaluate all teachers in mathematics at least four times per year.

______ g) All college-bound pupils take at least three years of mathematics in grades 9-12.

______ h) All non-college-bound pupils take at least three years of mathematics in grades 9-12.

*Professional Standards for Selection and Implementation of Instructional Materials**

The content, organization, and style of instructional materials, particularly basal textbooks, are among the most important factors influencing outcomes of school mathematics programs. Thus, the process of selecting new textbooks and ancillary materials is a critical step in planning curricula and preparing for instruction. This process includes setting criteria, screening materials, making choices, and implementing the new program. Although teachers and students are vitally concerned with the quality of the instructional materials they must use, the selection and preparation for use of those materials are often constrained by inappropriate selection and implementation processes.

In order to improve curriculum and instruction in elementary and secondary mathematics programs, the National Council of Teachers of Mathematics has adopted the following standards for selecting instructional materials and implementing programs using those materials. The Council urges compliance with these standards by all school districts.

Standard 1: The Review and Selection Committee

> *The entire process of textbook selection should be led by teachers and supervisors with expertise and responsibility in mathematics education. Except in most unusual circumstances, the recommendations of the committee must be followed.*

Membership on the Review and Selection Committee should include, but not necessarily be limited to, (1) classroom teachers representing the grades or courses for which instructional materials are being selected as well as teachers in grades or courses that precede or follow the target courses; (2) mathematicians or mathematics educators with knowledge of content, curriculum, and pedagogical trends covering the full spectrum of K-12 mathematics and beyond; and (3) school administrators who have overall responsibility for supporting and guiding the implementation of new programs.

Selection of committee members must be made in consultation with persons having expertise and responsibility in mathematics education. All persons selected should be full working members. Where the talents of a school or district mathematics staff cannot fulfill the roles described above, appropriate outside consultants should be obtained.

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* *This is an official position statement of the National Council of Teachers of Mathematics. It is reprinted with permission.*

Standard 2: Criteria for Selection of Instructional Materials

Criteria for selection must be defined and made public prior to the review process.

The criteria should identify the desired instructional approach as well as the mathematics content. Criteria for selection should be based on program needs, which can be identified by:

- evaluation of the present program, including analysis of student achievement, attitudes, and learning characteristics and a survey of teacher opinions;
- current research on the teaching and learning of mathematics;
- recommendations of professional organizations;
- curriculum guidelines of local, state or provincial, and national governments.

Difficulties in implementation should be considered only if they seem insurmountable. Any adoption of new instructional materials will create some problems of implementation. However, the committee's goal should be the identification of the best possible program, not the easiest to implement.

Difficulties in implementation should be considered only if they seem insurmountable.

Copyright dates and formula-determined reading levels are generally inappropriate as selection criteria. Although fashions in content and pedagogy change, a newer (or older) copyright does not assure higher quality. Because of the large technical vocabulary associated with mathematics, results of standard readability formulas should be used with extreme caution.

Standard 3: The Review of Instructional Materials

The Review and Selection Committee should actively seek materials from a variety of sources and evaluate the materials on the basis of the previously defined criteria.

The process of identifying, collecting, and examining material for potential adoption takes time to be done well. Time for committee members to do their work must be made available by the school system.

To identify appropriate instructional materials, the committee should consult:

- recommendations of teachers and supervisor in the district and in nearby districts having similar characteristics and goals;
- recommendations or authorizations of state or provincial committees where they exist;
- critical reviews in professional journals;
- independent information bases, such as the National Diffusion Network;
- representatives of all publishers of appropriate material.

Instructional materials developed by nonprofit organizations as well as commercial publishers should be considered. Whenever possible, committee representatives should examine materials displayed at a national or regional NCTM meeting. Interviews with members of the author team or a publisher's representative may also be helpful.

Once the materials for review have been collected, each should be evaluated for the:

- content coverage and development of each major curricular strand;
- development of the full range of cognitive abilities, especially problem solving and analysis;
- correlation of the content with local, state or provincial, or national scope-and-sequence guidelines;
- correlation of the content and instructional approach with research on mathematics teaching and learning;
- correlation of the content and instructional approach with the standards of professional mathematics organizations;
- correlation between textual features and characteristics of special students;
- evidence of success based on publishers' field-validation studies, implementation in other districts, critical reviews in professional journals and reports, or local field testing where possible;
- representation of the diversity of national social, political, and economic life.

Potential financial savings do not justify the selection of less desirable materials.

Since the cost of instructional materials is only a small fraction of any school's budget, the cost of these materials should not be a factor. Potential financial savings do not justify the selection of less desirable materials. Where textbooks are accompanied by ancillary materials, however, it is important to assess the extent to which successful use of the text depends on such extra materials.

Standard 4: Implementation of New Instructional Materials

> *The selection of new instructional materials must be only part of a careful curriculum-development process that includes appropriate inservice preparation for teachers and supervisors and a formal evaluation of the new program's effectiveness.*

Inservice preparation should acquaint all teachers with the general program goals and the emphasis for each topic in every course.

Inservice preparation should acquaint all teachers with the general program goals and the emphasis for each topic in every course. This emphasis should, where possible, be identified in terms of the appropriate number of instructional days to be devoted to each topic, and the topics should be correlated with specific portions of the new instructional materials. Inservice preparation should also provide any necessary additional mathematics background as well as preferred instructional approaches to the mathematics content.

Implementation should include periodic evaluation surveys of teachers' and students' reactions to the materials as well as an analysis of student achievement. Results of these evaluations should be used to evaluate the achievement of the program goals and to plan for ongoing modification of the program. Implementation is typically more successful when such inservice support continues throughout the academic year.

Standard 5: Professional Integrity

The instructional materials selection process must not allow the professional integrity of the process to be compromised.

Any person connected with the process of selecting and implementing instructional materials who has a financial interest in any program being considered must make that interest known publicly at the outset. For the duration of the selection process those involved must not accept gifts of travel, entertainment, or materials from a vendor.

The instructional materials recommended by the Review and Selection Committee must be adopted unless irregularities in the selection process are documented.

Textbook Selection

The following criteria on table 15 are suggested for the evaluation of textbooks for grades K-8. Districts should review and modify the criteria to fit their own curriculum priorities. For another set of criteria see *How To Evaluate Mathematics Textbooks*, a publication of the National Council of Teachers of Mathematics.

A textbook adoption committee should review and revise the criteria checklist before using it as a basis for textbook selection. Another activity that should take place prior to the application of the criteria to the textbooks being considered is a discussion of the interpretation of the scoring for each textbook series. It would be a mistake to automatically select the text with the highest total score since no attempt has been made to balance the list. It is not the case that each suggested criterion is equally as important as all others. Each evaluator should probably attempt to give each textbook a composite score in each of the categories: philosophy, methodology, supplementary aids, and mathematical content—problem solving, measurement, computation, geometry, statistics, and calculators and computers. A low score in any one of these categories is probably sufficient to eliminate a program from consideration.

Table 15

Textbook Selection Criteria

Scale: **1** (Inadequate), **2** (Marginal), **3** (Adequate), **4** (Excellent)

Philosophy of Program

The program

______ a) is organized around problem solving and integrates problem solving into all activities at all grade levels.

______ b) stresses the applications of mathematics in everyday situations.

______ c) stresses estimation and approximation in mental calculation and checking the reasonableness of all results.

______ d) introduces concepts using appropriate concrete examples and makes a transition through pictorial representations to abstract representations and processes as appropriate.

______ e) uses a spiral development.

______ f) uses calculators and computers to develop mathematical concepts as appropriate.

______ g) considers the mathematical needs of the students as adults in the twenty-first century.

Methodology

The program provides

______ a) mathematical concepts correctly and consistently used in explanations.

______ b) a variety of approaches in developing a topic.

______ c) a basic skills development and maintenance program.

______ d) adequate material for written assignments and oral exercises.

______ e) an emphasis on estimation and "guess-and-test" techniques in problem solving.

______ f) mental arithmetic activities.

______ g) independent thinking activities.

______ h) exercises graduated in difficulty.

______ i) consideration of individual differences.

______ j) clarity of examples.

______ k) supplementary activities for high achievers.

______ l) supplementary remedial activities for low achievers.

Supplementary Aids

______ a) The teacher's manual contains mathematical background for content of program, rationale for inclusion of topics, suggestions for teaching, and suggestions for enrichment activities.

______ b) A pupil evaluation program based on criterion referenced testing keyed to program objectives is available.

______ c) Computer software is available for pupil drill and practice, concept development, pupil testing, and record keeping.

______ d) Manipulative materials for use with program activities are available.

______ e) The publishing company provides consultants, films, or other inservice aids.

Mathematical Content

Problem Solving

The program

______ a) provides continuous emphasis on problem solving at all grade levels.

______ b) develops recognition of appropriate algorithms for applying mathematics to everyday situations.

______ c) develops a variety of techniques for solving problems having no standard algorithm.

______ d) includes problems with more than one correct answer.

Measurement

The program

______ a) develops measurement concepts through the measuring of objects and quantities.

______ b) provides opportunities to measure time and money.

______ c) develops and applies both nonstandard and standard units for length, area, volume, capacity, and angle measure, including metric units where appropriate.

______ d) provides experiences with approximate measures.

______ e) develops a knowledge of equivalent measures.

______ f) creates an awareness of the approximate nature of measurement and the error inherent in the measurement process.

______ g) develops area formulas for squares, rectangles, parallelograms, and triangles.

Computation

The program

______ a) develops an understanding of numbers and numeration, including place value.

______ b) stresses the meaning of the operations of addition, subtraction, multiplication, and division as applied to whole numbers and fractions.

______ c) develops computational algorithms through concrete and pictorial procedures.

______ d) uses computational skills appropriately in problem solving, including applying mathematics to everyday situations.

______ e) emphasizes estimation and alertness to the reasonableness of results.

______ f) develops the ability to use mental computation, estimation, pencil and paper computation, and computation with aids such as tables, calculators, and computers, and develops the ability to determine which method of computation is appropriate for a given situation.

______ g) emphasizes the inverse relationship between addition and subtraction and between multiplication and division.

______ h) provides opportunities for periodic review of previously learned skills.

Geometry

The program

______ a) develops recognition of common plane and solid figures.

______ b) contains activities leading to the identification of properties of common plane and solid figures.

______ c) provides manipulative and conceptual experiences with points, lines, and planes at appropriate grade levels.

______ d) develops recognition of parallel and perpendicular lines and planes.

______ e) includes activities which develop symmetry concepts and recognition of symmetry in plane and solid figures.

______ f) includes activities which develop congruence concepts, including congruence of segments, angles, polygons, and solid figures.

______ g) includes activities which develop similarity concepts and recognition of similarity in plane and solid figures.

Statistics

The program

______ a) introduces graph building and graph reading at the beginning (grade K or 1) and continues throughout (using pictographs, bar graphs, circle graphs, or broken line graphs as appropriate).

______ b) includes activities which require reading, interpreting, and constructing tables and charts for the communication of information.

______ c) provides activities with numbers on a number line, and extends to two dimensional and three dimensional coordinate systems.

______ d) develops concepts of probabilities of simple events.

______ e) provides activities in which the probability of an event is estimated from statistical data.

______ f) introduces common measures of dispersion and central tendency (range, mean, median, mode) and their value in decision making.

Calculators and Computers

The program

______ a) includes calculator activities at all grade levels.

______ b) includes computer concepts necessary to use the computer as a tool in mathematical problem solving.

______ c) uses computer activities to develop mathematics concepts.

______ d) provides appropriate opportunities to use calculators and computers in problem solving.

References

The Arithmetic Teacher. For teachers of mathematics in grades K-8. Published ten times per year by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091.

Comprehensive School Mathematics Program (CSMP). A curriculum for students in grades K-6. Developed at the Central Midwest Regional Education Laboratory (CEMREL) and now distributed by McREL, 4709 Belleview Avenue, Kansas City, MO 64112.

Developing Mathematical Processes (DMP). A curriculum for students in grades K-6 developed at the University of Wisconsin-Madison, Research and Development Center and now distributed by Delta Education, P.O. Box M, Nashua, NH 03061.

The Mathematics Teacher. For teachers of mathematics in grades 7-12. Published ten times per year by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091.

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Issues in Mathematics Education

11

Equity
Instructional Issues
Calculators
Instructional Time
Individual Differences
Teacher Education and Certification
Pupil Evaluation
Reading and Mathematics
Environmental Education and Mathematics
Career Education and Mathematics
Computers and Mathematics

Equity

The state and the nation recognize the differences in the experiences of women and of men, of all races, colors, and ethnic groups and of people of varied physical and mental abilities. These factors often result in the sorting, grouping, and tracking of minority, female, and disabled students in stereotyped patterns that prevent them from exploring all options and opportunities according to their individual talents and interests. The cost of bias to academic achievement, psychological and physical development, careers and family relationships is significant. Each student should have the opportunity to observe his or her own place in the curriculum, to grow and develop, and to attain identity.

To that end, the Department of Public Instruction recommends the inclusion of all groups in the curriculum as well as their equitable representation in teaching materials. The following forms of bias should be eliminated.

Invisibility: underrepresentation of certain groups, which leads to the implication that these groups are of less value, importance, and significance

Stereotyping: assigning only traditional and rigid roles or attributes to a group, thus limiting the abilities and potential of that group; denying students a knowledge of the diversity, complexity, and variations of any group of individuals

Imbalance/selectivity: presenting only one interpretation of an issue, situation, or group; distorting reality and ignoring complex and differing viewpoints through selective presentation of materials

Unreality: presenting an unrealistic portrayal of history and contemporary life experience

Fragmentation/isolation: separating issues related to minorities and women from the main body of the text

Linguistic bias: excluding role and importance of females by constant use of the generic "he" and sex-biased words

Equity in mathematics programs should be measured both by enrollments and by student achievement. In Wisconsin, on the average, enrollments of males and females are equitable through second year algebra in eleventh grade. Enrollments of males and females are not equitable in fourth and fifth year courses. Enrollments based on socio-economic status are, in general, not equitable beginning in the ninth grade.

Measures of performance on tests of mathematical achievement sometimes show differences in means between groups of different sexes, races, or socio-economic status. While these figures do provide evidence of difference in achievement, they do not provide evidence of difference in ability.

The importance of mathematics to an individual is independent of that individual's sex, ethnic origin, and socio-economic status. Furthermore, there is no evidence that the ability to learn mathematics is genetically related to sex, ethnic origin, or socio-economic status. If any such differences are discovered in an evaluation of the school mathematics program, the district must work aggressively to eliminate them. An exemplary program will not have such differences.

The importance of mathematics to an individual is independent of that individual's sex, ethnic origin, and socio-economic status.

Instructional Issues

Mastery Learning and the Spiral Curriculum

In order for a curriculum design based either on mastery learning or on the spiral curriculum to be effectively implemented, common misinterpretations must be avoided. All children should be expected to learn something as a result of each encounter with mathematics. However, it is not reasonable to expect that all children will learn, and master, all the mathematics discussed in any one lesson. Advocates of mastery learning should not insist on a level of mastery which is unattainable. This diminishes student interest in mathematics. Furthermore, if student progress is blocked due to lack of mastery of one topic, the opportunity to be successful on other topics is denied. Children can study computational algorithms before they have mastered all the facts. They can then continue to learn facts in the context of the algorithms. They can also study geometry before mastering arithmetic. Children who have not mastered the factoring of trinomials can still learn a lot of algebra.

On the other hand, advocates of a spiral curriculum sometimes fail to require any learning, declaring that this topic will come up again and that the child can learn it later. By this rationale, learning can be deferred indefinitely, and the child may never learn the expected content.

The appropriate resolution is to be very clear which part of the daily lesson is to be understood by the students, specifically, the part necessary to do the homework successfully. During the instruction phase of the lesson the teacher should work on the development of those objectives, and during the assessing comprehension phase the teacher should determine that most students will be able to complete the assignment successfully. For a complete discussion of the phases of a lesson, see *Effective Teaching of Mathematics* (Chambers, 1984).

. . . the principle of spaced review . . . should be an integral part of all mathematics programs.

Whatever plan is adopted, the principle of spaced review is firmly established in research and should be an integral part of all mathematics programs. This principle recognizes that mastery, once attained, is not retained without review on a regular basis. It also recognizes the value of short daily reviews rather than full period reviews on the day before a test. Incorporating earlier material into daily review sessions, daily homework assignments, and tests is a good way to implement the spaced review concept.

Mathematical Symbols and Symbol Manipulation

Mathematics has achieved much of its power through its deductive methodology and the emergence of a symbol system which allows very complex ideas to be stated in a very abbreviated form. These features greatly facilitate the communication of mathematical ideas among experts. At the same time, they are frequently responsible for the difficulties encountered in communicating mathematical ideas to novices. Since most students in elementary and secondary schools are novices, the use of mathematical symbolism and deduction must be treated with great care. It is important at every stage of mathematics instruction that teachers and students use symbols that students understand. It is also important that teachers not employ deduction more formally than students can understand.

. . . the use of mathematical symbolism and deduction must be treated with great care.

The same symbol can be thought of as concrete or abstract, depending on the maturity of the student. Among first graders "387" is an abstract representation of a large collection of objects. Among ninth grade algebra students, "387" can be used as a concrete representation of the polynomial "$3x^2 + 8x + 7$" when $x = 10$, and properties of polynomials can be illustrated by using integers in expanded notation. Our goal is to arrive at the highest possible level of abstraction comprehensible by each student, thus providing the students greater access to the full power of mathematics. However, premature use of abstraction beyond the comprehension of the students is counterproductive.

One activity of mathematicians is to express collections of mathematical symbols in equivalent forms. Elementary school mathematicians express 6/8 as 3/4. High school mathematicians express $3x + 6 - 14 = x - 2$ as $x = 3$. Many rules are introduced for transforming one expression into another, and students spend much of their time learning and practicing these rules.

New computer technology is forcing a reevaluation of the role of symbol manipulation in elementary and secondary mathematics. Small, inexpensive calculators can be used for much of the symbol manipulation ordinarily taught in the elementary grades. A prominent U.S. mathematician, Henry Pollak, has observed that "Two-thirds of all elementary school mathematics is taught in order to make calculators and microprocessors obsolete" (Romberg, 1984).

. . . "Two-thirds of all elementary school mathematics is taught in order to make calculators and microprocessors obsolete."

Symbol manipulation also prevails at the secondary level. Computer technology is now capable of providing compact, inexpensive symbol processors which "perform symbolic algebra and calculus as proficiently as the most able high school and undergraduate students" (Fey, 1984).

Curriculum planners are encouraged to consider two recent reports which question the current emphasis on symbol manipulation in school mathematics. *The Impact of Computing Technology on School Mathematics* is included in Appendix G. *Computing & Mathematics: The*

Impact on Secondary School Curricula, edited by James T. Fey and published by the National Council of Teachers of Mathematics, is an important resource in considering the future direction of the secondary school curriculum.

The role of computer technology notwithstanding, teachers at all grade levels are encouraged to introduce symbolism only when a concept has been introduced, studied, and practiced, and the introduction of the symbolism comes as a convenience, not as an obstacle. The student is then able to use the symbolism to consolidate the new knowledge with the old and move on to other new ideas.

Effective Teaching Research

While an improved curriculum is an important step toward improving school mathematics programs, improved instruction also plays an important role—perhaps the most important role. According to former U.S. Commissioner of Education Harold Howe II, "The improvement of pedagogy is the most likely source of a real solution to the shortcomings of America's schools" (Gage, 1984).

. . . "The improvement of pedagogy is the most likely source of a real solution to the shortcomings of America's schools."

Recent research offers many suggestions for improving classroom teaching. Generally researchers note that effective teaching results from combinations of teacher behaviors. Single variables are not sufficient to account for differences in pupil achievement. With this in mind, teachers are encouraged to incorporate as many of the recommended techniques as possible.

In a study of fourth grade teachers which has subsequently been replicated at other grade levels, Good and Grouws (1983) noted that effective teachers:

- taught the class as a group;
- presented information actively;
- presented information clearly;
- were non-evaluative;
- created a relaxed environment;
- expected higher achievement;
- gave more homework;
- used a faster pace;
- had fewer behavioral problems.

The instructional strategies and classroom management patterns identified as effective by Good and Grouws are described in detail in *Active Mathematics Teaching* (1983) and *Effective Teaching of Mathematics* (Chambers, 1984). A sample structure for a daily lesson is given in this guide on p. 128.

In a study of teachers in seventh and eighth grade classrooms, Evertson, Emmer, and Brophy (1980) found that effective teachers:

- devoted 50 percent of class time to active instruction;
- asked more process and product questions;
- asked more follow-up questions;
- provided more encouragement;
- engaged in more problem-solving behavior;

- had less transition time;
- had less group discipline time;
- presented information more clearly;
- asked more higher cognitive level questions;
- were more effective classroom managers.

The instructional model advocated by Madeline Hunter is similar to that of Good and Grouws. It claims to be a generic model which can be used for the instruction of any discipline. While there is a strong research base for the components of the Madeline Hunter model, there are at the present time no research studies which validate the model in mathematics classrooms.

There is some research support for small group instruction in mathematics. Gersten and Carnine (1984) conducted a study in first through third grade classrooms with large populations of students from low-income families. They used structured teaching of small groups of six to ten students. Principles of mastery learning were used, and students received at least 30 minutes per day of intensive instruction. Unlike the programs mentioned above which were independent of the instructional materials used, this study involves the use of particular instructional materials, the scripted DISTAR materials. Like the others, it involves a rapid pacing of tasks and practices designed to maximize academic learning time (the time each student is successfully engaged in mathematics). Appropriate correction of student errors, continual assessment of student progress, and continual assessment of teacher performance are other features of the program.

Gersten and Carnine found that their low-IQ students gained 1.0 grades per year and that the mean performance of the low-income students was equivalent to that of their middle-income peers.

Another study, involving small groups (Slavin, 1984), used a pattern called Team Assisted Individualization in grades 4 through 6. Students were organized into four-to-five member teams of mixed ability. The students used individualized curriculum materials and cooperated with seatwork, scoring, and explaining in their groups. Incentives were provided by a weekly competition among the teams for awards. The teacher worked daily with three teaching groups of students working on the same level.

A reasonable combination of whole-group instruction and small group work would be to have whole-group instruction, done according to the Good and Grouws model, followed by seatwork done in mixed-ability groups of four students.

Calculators

Generally speaking, anything which makes mathematics easier should be welcomed, praised, and exploited. By fully exploiting calculators in mathematics instruction we can reallocate some instructional time to those aspects of mathematics for which calculators are not helpful. With certain caveats which will be mentioned here, teachers and students should have unrestricted access to calculators in class, for homework, and for tests.

This does not mean that all of computational arithmetic should be abandoned. In order to efficiently use calculators, users must possess certain mathematical skills including the following.

- Knowledge of number facts.

Without knowledge of number facts, operators will be unable to determine whether their results are reasonable.

- Proficiency with computational algorithms for small numbers.

Efficient use of calculators includes the ability to use another method when that method is more efficient. Some computations are more efficiently performed with pencil and paper or mental arithmetic than with calculators. The Mathematics Curriculum Task Force recommends the following levels of computational proficiency with pencil and paper algorithms: the ability to add and subtract three-digit numbers, the ability to multiply two two-digit numbers, and the ability to divide by one-digit numbers and one-digit multiples of ten, one hundred, and one thousand.

- Proficiency in mental arithmetic.

Estimation using mental arithmetic with numbers rounded to one significant digit is a good way to check the reasonableness of a computation done with a calculator.

- Knowledge of the calculator's memory features and order of operations conventions.

When a series of operations is to be carried out, it may be necessary to do a partial computation and store the result. Otherwise, it may be desirable to change to an equivalent expression. For example:

$$3176 + (24985 \div 764) \rightarrow (24985 \div 764) + 3176.$$

By fully exploiting calculators in mathematics instruction we can reallocate some instructional time to those aspects of mathematics for which calculators are not helpful.

Efficient use of calculators includes the ability to use another method when that method is more efficient.

The objections to the use of calculators can be addressed by knowledgeable curriculum planners.

Objection: Children who use calculators will become dependent on them. (Their minds will turn to Jello.)
Response: Research indicates that children who use calculators on a regular basis do no worse on pencil and paper arithmetic than children who do not use calculators.

Objection: Using calculators deprives children of opportunities to exercise their brains by doing difficult computations.
Response: There is no evidence that practice with difficult computations produces other benefits, such as improved "thinking," or that it transfers to other goals of the school mathematics program.

Objection: What will the child do if the batteries go dead?
Response: There are many possible responses. First and foremost, the calculator is a convenience, not a necessity. Properly educated children will not become mathematical cripples without their calculators. Most will be able to use alternate strategies to perform the computation. Second, solar powered calculators are now readily available at a cost of

less than five dollars. With these, the life of the battery is not an issue. Third, if necessary, get new batteries or a new calculator (or borrow one). One might more reasonably ask, in the case of a child who knows only paper-and-pencil methods, "What will the child do if the pencil lead breaks?"

Objection: What will the child do if he or she forgets his or her calculator?
Response: The first response to the above objection applies here. Furthermore, the child is no more likely to forget a calculator than a pencil or paper.

Objection: Calculators are too expensive for some children.
Response: A solar powered four-function calculator with memory can be purchased for less than five dollars. This is less than most children spend on pencils and paper. If some children cannot provide their own, the schools should keep some on hand.

Objection: The ability to purchase a more elaborate calculator may give some children an advantage.
Response: Grading systems and evaluation procedures can be devised which will not penalize children because they don't have the most expensive equipment.

There are specific situations in which it is appropriate to prohibit students from using calculators.

There are specific situations in which it is appropriate to prohibit students from using calculators. Students should learn basic facts and algorithms. When drilling on basic facts or algorithms, calculators should not be used. When evaluating knowledge of basic facts or algorithms, calculators should not be used. When teaching estimation skills and mental arithmetic, calculators should not be used.

As part of its mathematics curriculum development process, a district should develop a policy on the use of calculators in the classroom. The following policy statement, developed by the National Council of Teachers of Mathematics, endorses the use of calculators in the classroom and provides some language which districts may want to include in their own policy.

Calculators in the Classroom

The National Council of Teachers of Mathematics encourages the use of calculators in the classroom as instructional aids and computational tools. Calculators give mathematics educators new opportunities to help their students learn mathematics and solve contemporary problems. The use of calculators, however, will not replace the necessity of learning computational skills.

As instructional aids, calculators can support the development and discovery of mathematical concepts. As computational tools, they reduce the time needed to solve problems, thereby allowing the consideration of a wider variety of applications. Furthermore, the use of calculators requires students to focus on the analysis of problems and the selection of appropriate operations. The effective use of calculators can improve student attitudes toward, and increase interest in mathematics.

Other electronic devices, programmed to generate questions and activities, that provide immediate feedback to students, are not to be confused with calculators or computers. These devices can be used to reinforce computational skills through drill.

The NCTM Instructional Affairs Committee has identified the following justifications for using the calculator in the schools.

- The calculator can be used to encourage students to be inquisitive and creative as they experiment with mathematical ideas.
- The calculator can be used to assist the individual to become a wiser consumer.
- The calculator can be used to reinforce the learning of the basic number facts and properties in addition, subtraction, multiplication, and division.
- The calculator can be used to develop the understanding of computational algorithms by repeated operations.
- The calculator can be used to serve as a flexible "answer key" to verify the results of computation.
- The calculator can be used as a resource tool that promotes student independence in problem solving.
- The calculator can be used to solve problems that previously have been too time-consuming or impractical to be done with paper and pencil.
- The calculator can be used to formulate generalizations from patterns of numbers that are displayed.
- The calculator can be used to decrease the time needed to solve difficult computations.

For further recommendations on the uses of computing technology in mathematics instruction, see *An Agenda for Action* (Appendix C), and *The Impact of Computing Technology on School Mathematics* (Appendix G).

Instructional Time

The Mathematics Curriculum Task Force recommends 50 minutes of instruction in mathematics per day. This recommendation, endorsed by the Wisconsin Department of Public Instruction, is somewhat less than that recommended in other reports. The National Commission on Excellence in Education recommends "that significantly more time be devoted to learning the New Basics. This will require more effective use of the existing school day, a longer school day, or a lengthened school year" (the Commission identifies three years of mathematics in its definition of the the Five New Basics). The Commission goes on to recommend "7-hour school days, as well as a 200- to 220-day school year."

The Mathematics Curriculum Task Force recommends 50 minutes of instruction in mathematics per day.

The National Council of Teachers of Mathematics recommends that three years of mathematics be required of all students and that time allocated to mathematics "range from a minimum of five hours a week in the primary grades to a minimum of about seven hours a week in the upper grades." The National Science Board Commission on Precollege Education in Mathematics, Science and Technology advocates "a minimum of 60 minutes per day of mathematics" and at least three years of mathematics in high school for all students. With these recommendations in mind, 50 minutes per day should be viewed as a necessary, although not sufficient condition for good mathematics instruction.

State law now requires that students in grades K through 6 receive at least 1,050 hours of instruction per year. Evenly distributed over the mandatory 180 school days, this would result in 1,750 instructional minutes per week. Table 16 (Minimum Allocated Instructional Time) suggests a division of that instructional time to assure an equitable distribution to all subject areas.

Wisconsin statutes require students in grades seven through twelve to receive 1,137 hours of instruction per year. Based on a school year of 180 days this would permit seven 50-minute periods per day. Schools offering only the required number of hours of instruction scheduled in eight-period days could schedule only 42- or 43-minute classes (allowing 4 to 5 minutes between periods).

Students in 50-minute class periods will, over a period of two years, receive one semester more instruction than students in 40-minute classes. In order to graduate, students are now required to enroll in mathematics at least ten years. Over a period of ten years, students in 50-minute classes receive two and one-half years more instruction in mathematics than students in 40-minute periods.

Students in 50-minute class periods will, over a period of two years, receive one semester more instruction than students in 40-minute classes.

The length of the instructional period is not the only important time-related variable. In one study of mathematics classes students were observed to be on task for about 75 percent of the instructional period. Teachers are encouraged to maximize the amount of time students are actually engaged in learning mathematics.

One model for maximizing student engaged time is the following:

Daily Review (about 10 minutes total)
  Review previous day's lesson (*2-3 minutes*)
  Collect and deal with homework (*4-5 minutes*)
  Mental computation (*2-4 minutes*)
Development (about 25 minutes total)
  Review of prerequisite skills and concepts (*3-5 minutes*)
  Develop new material meaningfully (*at least 10 minutes*)
  Assess student comprehension (*3-5 minutes*)
  Repeat and elaborate on meaning portion (*as necessary*)
Seatwork (about 15 minutes total)
  Alert students that work will be checked
  Provide uninterrupted, successful practice
  Check students' work
Homework Assignment (1-2 minutes)

When students are not merely engaged in mathematics, but successfully engaged, their time is even more productive. Skilled teachers will find ways to make the instruction and seatwork of an appropriate level of difficulty so that students can maximize their successfully engaged time each day.

Minimum Allocated Instructional Time: Recommended by the Wisconsin Department of Public Instruction*

Grades K-6: minutes per week for a six-hour instructional day

Grade Level	K**	1	2	3	4	5	6
Reading/English Language Arts***	30%	700	700	600	600	500	425
Mathematics	10%	250	250	250	250	250	250
Social Studies	10%	125	150	175	200	225	250
Science	10%	100	100	150	150	175	250
Health	10%	75	75	100	100	125	125
Physical Education	10%	150	150	150	150	150	150
Art	10%	90	90	90	90	90	90
Music	10%	75	75	75	75	75	75
Foreign Language	---	---	---	---	---	100	100
Environmental Education****	****	****	****	****	****	****	****
Total Allocated Instructional Minutes		1565	1590	1590	1615	1690	1715

* While there is a recommended allocated instructional time for each subject, educators are encouraged to integrate subjects within the curriculum whenever possible.

** Up to one-third of each day in the kindergarten schedule may be reserved for students' self-selected instructional activities. The allocated instructional time recommendations presented in column K apply only to the portion of the schedule planned for teacher-directed activities. The time allocations for kindergarten are expressed in percentages to facilitate planning for various kindergarten schedules.

*** Instruction specifically designed to strengthen reading and writing abilities should be integrated with other disciplines, e.g., health, science, and social studies. For further discussion of the recommendations, see *A Guide to Curriculum Planning in Reading* and *A Guide to Curriculum Planning in English Language Arts*.

**** Environmental topics should be infused into instructional units in all subjects, with the greatest emphasis in health, science, and social studies. A significant amount of time should be devoted to such topics. For a further discussion of this recommendation, see *A Guide to Curriculum Planning in Environmental Education*.

Acceleration/Enrichment

All schools have at least one student who seems to learn so much more rapidly than the others that it is difficult for the teacher to accommodate that student within the context of whole class instruction. Teachers frequently accommodate these students by accelerating the students through the text and then into text material for subsequent years. This solution is convenient for the teacher but is not generally advantageous for the student. Furthermore, rather than directly addressing the problem, the problem is merely transferred to the next teacher. At some time the problem will have to be addressed directly.

The problems created for students are identified by the National Council of Teachers of Mathematics in the following position statement on vertical acceleration.

This statement is an official NCTM position. It was stimulated by a Delegate Assembly resolution, developed by the Council's Instructional Issues Advisory Committee, and adopted by the Board of Directors.

*Vertical Acceleration Position Statement**

The decision to accelerate students vertically through grades K-12 is a critical one and should be made in consultation with the student, parent, teacher, counselor, administrator, and the student's academic record. Vertical acceleration should be implemented *only* for the extremely talented and productive student.

In almost all cases, a student would benefit more from a program of horizontal acceleration (enrichment) than from one of vertical acceleration. *An Agenda for Action* makes this point in stating that:

In almost all cases, a student would benefit more from a program of horizontal acceleration (enrichment) than from one of vertical acceleration.

> in general, programs for the gifted student should be based on a sequential program of enrichment through ingenious problem-solving opportunities rather than through acceleration alone.

Among the topics that should be included in all mathematics programs but that also suggest important areas for an enriched curriculum are the following.

- Calculators and computers
- Problem posing and solving
- Development of facility with mathematical language
- Mental arithmetic
- Estimation and approximation
- Probability and statistics
- Intuitive geometry

** National Council of Teachers of Mathematics position statement.*

An analysis of student abilities and of the mathematical applications in current careers suggests that programs of acceleration must consider the role of discrete mathematics (combinatorics, graph theory, and probability) and elementary computing skills, as well as the added work of setting up and planning the calculation of answers to complex problems based on algebraic concepts and principles studied in the secondary curriculum. Stressing the development of algorithms for complex calculations can serve to tie the traditional secondary curriculum to the emerging topics listed above and help build a bridge to the collegiate study of mathematics.

The preparation of teachers of accelerated courses should be in accordance with the *Guidelines for the Preparation of Teachers of Mathematics*. These teachers should be chosen from among a district's most motivating and talented instructors. They should have the ability to challenge student thought, guide student inquiry, and make appropriate suggestions for further independent study. In addition, such teachers should be provided with support services so that the school district can maximize its gifted students' mathematical potentials.

The records of accelerated students clearly show that a significant number discontinue the study of mathematics before they complete four years of high school mathematics. Of those who do remain in mathematics until graduation, many experience extreme difficulty with the subject. Therefore, the NCTM recommends the following.

> That vertical acceleration be considered only for a limited number of highly talented and mathematically creative students whose interest and attitudes clearly indicate that they have the ability and perseverance to complete a carefully designed sequential curriculum. For all but this select group, a strong, expanded program emphasizing mathematics enrichment is preferable.

(April 1983)

Homogeneous/Heterogeneous Grouping

Homogeneous grouping of students is a technique commonly used to enable teachers to accommodate individual differences. There are two serious problems associated with the practice of homogeneous grouping which should be considered before such a decision is made.

Teachers of homogeneously grouped classes frequently make the mistake of concluding that since their students have been grouped according to ability, there are no differences within groups. They may assume that if differences exist they are small. In either case the teacher has a convenient reason not to make an extra effort to accommodate individual differences in students. No matter how large a school is, extensive differences will still exist among students grouped homogeneously by ability. The problem is even more serious when students are grouped according to a non-mathematical measure of ability such as reading level or comprehensive test score which includes mathematics but includes other domains as well.

. . . ability grouping appears to deprive students in the lower groups of an equal educational opportunity.

Perhaps more seriously, ability grouping appears to deprive students in the lower groups of an equal educational opportunity. According to Maureen Hallinan (1984) of the University of Notre Dame, "In general, students in low grades and ability groups are given fewer and poorer opportunities to learn than their peers in higher level groups." She observed that in low ranked groups more time is given to administration and discipline and less to teaching. Teachers spend less time on preparation, do a poorer job of teaching, present material at a slower pace, and use less interesting and less challenging material.

Thomas Good and Susan Marshall (1984) observe "that tracking and ability grouping have few desirable consequences for low-ability students. Research indicates that in many classrooms teachers err by holding expectations that are too low, by pacing instruction too slowly, and by ignoring or under-emphasizing the substantive aspects of tasks when instructing low groups."

Teacher Education and Certification

Wisconsin law requires all teachers in public schools to be certified. To be certified to teach mathematics in grades 9-12, a teacher must complete a mathematics minor of at least 22 credits. The minor must be approved by the Department of Public Instruction. Coursework in methods of teaching mathematics, and student teaching in mathematics are required of persons newly certified as of 1986. Permanent certification is no longer available, so teaching certificates are issued for five-year periods. To renew a certificate a teacher must complete six semester credits or the equivalent. These credits need not be in mathematics or mathematics-related coursework.

Most mathematics teachers in Wisconsin high schools have completed a major in mathematics. A major consists of a minimum of 34 credits. All persons certified by completing a major have also completed a course in methods of teaching mathematics and have done student teaching in mathematics. All major programs as well as all minor programs have been approved by the Department of Public Instruction.

Teachers of mathematics in grades eight and below are not required by law to have any training in mathematics or the teaching of mathematics.

Teachers of mathematics in grades 8 and below are not required by law to have any training in mathematics or the teaching of mathematics. Most programs approved to certify teachers in these grades require one course in mathematics and some methods of teaching mathematics as part of the program.

Persons certified to teach in grades 9-12 are almost always certified to teach in grades 7 and 8 also. However, due to shifting enrollment patterns and shifting building patterns over the last decade, along with a shortage of secondary certified mathematics teachers, many certified mathematics teachers in grades 7 and 8 have been reassigned to grades 9-12. Consequently, much of the mathematics teaching in grades 7 and 8 is now being done by elementary certified teachers.

Each year about 30 persons are issued emergency licenses to teach mathematics. These licenses are issued at the request of a district which attests that it is unable to hire a properly certified mathematics teacher. A teacher licensed to teach mathematics in this manner must complete six semester credits toward certification during the ensuing academic year or is not eligible for renewal of the one-year emergency license. Only one percent of the 3,000 fully licensed teachers of mathematics are teaching with an emergency license.

To say that a person is licensed does not necessarily say that person is fully qualified. The National Council of Teachers of Mathematics has recommended standards for certification which are far more demanding than the Wisconsin law now requires. Its report, *Guidelines for the Preparation of Teachers of Mathematics*, presents a detailed list of teacher objectives at each of four levels: early childhood and primary grades (ages 4 to 8), upper elementary and middle school grades (ages 8 to 12), junior high school grades (ages 12 to 14), and senior high school grades (ages 14 to 18).

The recommendations for teachers of early childhood and primary school grades call for a minimum of two years of algebra and one year of geometry in high school followed by nine semester hours of instruction in mathematics and mathematics education in the undergraduate program. The recommendations for teachers of upper elementary and middle school grades call for an additional three semester hours in mathematics.

According to the Council, junior high school teachers should have four years of mathematics in high school and not less than 21 semester hours of mathematics and mathematics education as undergraduates, and senior high school teachers should have not less than 39 semester hours of mathematics and mathematics education.

Wisconsin has many fine teachers of mathematics at all grade levels. It is safe to say, however, that the certification requirements are considerably lower than the standards advocated by the National Council of Teachers of Mathematics.

In *Recommendations on the Mathematical Preparation of Teachers*, the Mathematical Association of America advocates even stricter requirements for certification of mathematics teachers.

In *School Mathematics: Options for the 1990s* (Romberg, 1984), a panel of distinguished mathematics educators has once again called for improved preparation of mathematics teachers, especially teachers of mathematics in grades 4-8. The panel recommends the following.

- In elementary school, specialist teachers of mathematics should teach all mathematics beginning no later than grade 4 and supervise mathematics instruction at earlier grade levels.
- In secondary schools, master teachers of mathematics should teach or supervise all mathematics instruction.
- A task force should be organized to propose certification standards for both elementary school specialist teachers of mathematics and secondary school master teachers of mathematics.

In elementary school, specialist teachers of mathematics should teach all mathematics beginning no later than grade 4 and supervise mathematics instruction at earlier grade levels.

Opponents of these proposals tend to cite administrative inconvenience and cost as the reasons that they cannot be implemented. The question is whether or not improved mathematics teaching and consequently improved mathematics learning are worth the inconvenience and cost.

Another very real objection is an even more practical one. No teacher education institution in Wisconsin currently has a program which is designed to prepare specialist teachers of mathematics in grades 4-8. This is probably due to the present lack of demand for such a program from students, which in turn is due to the certification requirements which allow persons to teach mathematics at those grade levels without special preparation.

These certification requirements will probably not be changed within the next five years. This means that initiatives for improvement in this area would have to come from local districts, and probably from mathematics committees in local districts. In addition to recommending changes in curriculum to improve the mathematics program, committees are encouraged to consider how changes in staffing patterns might also produce improvements.

While a shortage of certified mathematics teachers does not now exist in Wisconsin, the following position statement of the National Council of Teachers of Mathematics has some recommendations worth heeding.

The Mathematics Teacher Shortage and Retraining Programs[*]

The growing shortage of qualified teachers of mathematics in the United States has placed the quality of mathematics instruction for our youth in serious jeopardy. Immediate and concerted action is required to avoid the ominous consequences of a mathematically undertrained citizenry, unprepared for increasingly technological jobs. The severity of the situation for our society is apparent.

Many teachers without sufficient mathematics training are being asked to fill vacancies in mathematics classrooms. At the same time that society needs better-trained students in mathematics, the cadre of qualified and experienced teachers continues to diminish.

The repercussions of this situation are far reaching. National priorities of economic development, industrial productivity, and national defense all rely on a scientific and technological base with mathematics as its foundation. It is not an overstatement to suggest that the retention of our present economic, political, and military strength is dependent on our ability to produce mathematically capable students.

In confronting this issue, the National Council of Teachers of Mathematics takes the following positions.

- Students at all levels should be taught by fully qualified teachers whose mathematical backgrounds meet or exceed that outlined in the *Guidelines for the Preparation of Teachers of Mathematics*.
- Teachers of mathematics who have not met the competencies in the *Guidelines for the Preparation of Teachers of Mathematics* for their grade

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[*] *National Council of Teachers of Mathematics position statement.*

level should be working to remove any deficiencies at the earliest possible date.

> *Teachers of mathematics who have not met the competencies in the* Guidelines for the Preparation of Teachers of Mathematics *for their grade level should be working to remove any deficiencies at the earliest possible date.*

- Teachers of mathematics at all grade levels should work continually to upgrade their knowledge of mathematics and the skills for communicating it to students, and it is the responsibility of local school districts, in conjunction with other agencies and organizations, to provide professional development opportunities that foster such upgrading.
- The federal government, states, school districts, and institutions of higher education should cooperatively address the shortage of qualified mathematics teachers by establishing high-quality training and retraining programs based on the *Guidelines for the Preparation of Teachers of Mathematics* and by providing appropriate and sufficient financial support for these programs.
- School districts experiencing difficulties attracting and retraining qualified mathematics teachers should explore innovative strategies including bonuses, stipends, differential pay adjustments, step jumping, or business- and industry-sponsored fellowships in recognition of the growing disparity between salaries for mathematics teachers and salaries available in the private sector.
- School districts, teacher organizations, and local, state, and national mathematics organizations should bring the seriousness of the shortage of qualified mathematics teachers to the attention of parents, leaders, and policy makers at all levels and should be actively involved in formulating solutions to this problem.

Special Comments on Retraining Programs. In formulating the recommendations above, the National Council of Teachers of Mathematics is aware of the growing number of retraining programs aimed at assisting junior and senior high school teachers to obtain certification in mathematics. Unfortunately, many of these programs have set their requirements far below the standards suggested in the *Guidelines for the Preparation of Teachers of Mathematics*. Further, many suggest that the retraining necessary can be accomplished in exceedingly short periods of time and with limited attention to the need for relevant, up-to-date course content. In doing so, they fail both the teachers enrolled and the students these teachers, in turn, will instruct.

The National Council of Teachers of Mathematics is on record with respect to what constitutes an adequate level of preparation for teaching secondary school mathematics. We cannot support retraining programs that are not moving participants toward full compliance with the *Guidelines for the Preparation of Teachers of Mathematics*. Further, although the National Council of Teachers of Mathematics recognizes the emergency situation that exists in many school districts and is sympathetic with their plight, we cannot endorse the employment or use of faculty for secondary mathematics teaching who have not achieved at least the level of competency and associated course work suggested for junior high school teachers in the Guidelines.

> *. . . we cannot endorse the employment or use of faculty for secondary mathematics teaching who have not achieved at least the level of competency and associated course work suggested for junior high school teachers in the Guidelines.*

It is only through actions like those outlined above that our schools can effectively provide the high quality mathematics instruction needed by our young people and demanded by society.

(April 1983)

Pupil Evaluation

Pupil evaluation enables us to determine what a student knows or does not know so that instructional decisions can be made.

Pupil evaluation has become such a standard feature of mathematics instruction that teachers sometimes forget they do it. Then neither teachers or students are likely to benefit from the process.

Pupil evaluation serves three purposes. It enables teachers to assign to each pupil a letter or a number which claims to be a measure of the student's achievement. The compilation of those letters or numbers provides each student with an achievement history in the form of a transcript. This transcript may be used in applying for a job, for entrance to a college or other postsecondary education, or for other opportunities for which the student may be eligible.

Pupil evaluation enables teachers to determine what a student knows or does not know so that instructional decisions can be made. These decisions may involve diagnosing student errors, determining whether or not to move on to the next topic, or determining which topics should be introduced next.

Pupil evaluation enables a district to accumulate data for an evaluation of its instructional program. If pupil achievement records indicate that most students have learned most of the intended curriculum, the program may be thought of as successful. On the other hand, if large numbers of students have failed to learn some very important concepts, this reflects a weakness and should lead to decisions about changes in the instructional program.

For most classroom teachers, the most important purpose of pupil achievement should be providing data for subsequent instructional decisions. Unfortunately, many times instructional decisions are not based on the achievement data. Many times a teacher gives a chapter test, scores the papers, and enters letter grades in a grade book. The next day the papers are returned and discussed, and the next day's work routinely begins on the next chapter. Teachers always know which students are doing well and which students are not doing well, but don't always know the strengths and weaknesses of individual students.

Different methods are used to measure short term, intermediate term, and long term progress. Short term progress can be measured through the student's performance on the previous day's homework, written work in class, or responses in class to questions asked by the teacher. This information is an important indicator of the student's likely success later on, and provides an opportunity for immediate correction of errors or misunderstandings. When instruction is success-oriented, teachers should feel confident that when students leave the classroom they will be successful on the homework they will do that evening. To acquire that confidence the teacher will have to evaluate each student's progress on the material while the lesson is in progress.

Intermediate term progress is usually measured by teacher-made tests or departmental tests that cover a unit, chapter, or term. Teacher-made tests are usually objective-referenced. The teacher looks over the material from the unit or chapter, identifies the most important topics or concepts, and identifies questions or problems which measure the students' understanding. Rather than use an objective-referenced grading system, however, most teachers use a norm-referenced system in which all students are compared by means of a scale. Whether the teacher uses an absolute scale (93 - 100 = A; 85 - 92 = B; etc.) or grades "on the curve," the teacher ends up with a record of which students knew more and which students knew less, but has no record of what any student did or did not know. It is the absence of this objective-referenced data that prevents teachers from using the pupil evaluation for instructional decision making.

Long term evaluation may be achieved through final examinations which cover the work of the entire year, or through standardized achievement tests or objective referenced competency tests which cover the work of several years. Because of their infrequency, these tests seldom provide usable information for instructional decision making although they do provide data for the student's record and may also provide data which can be used for program evaluation.

For instructional purposes, the most important evaluation is the daily evaluation of each student. Teachers are encouraged to give more attention to this type of pupil evaluation and to find appropriate methods of recording the results of this evaluation and using this data to plan for instruction. For further suggestions, see *Effective Teaching of Mathematics* (Chambers, 1984).

Reading and Mathematics

Reading and mathematics are interrelated in two important ways. Mathematics provides a context for reading which requires different skills than reading stories or reading text material in other disciplines, social studies for example. Teachers of mathematics have an opportunity to provide instruction in those aspects of reading which are not emphasized otherwise. Second, skillful readers have opportunities to learn mathematics from print material. This provides a richer instructional environment than that in which students can only learn through instruction from the teacher.

Reading difficulties are often blamed for difficulties students have in solving story problems. A relatively small number of students do have difficulty reading the simple statements which are used to state story problems. For them, reading is a factor. Teachers can treat this problem in two ways. One is through reading instruction, which helps the students read and understand the problem situation. Another is to present the problem situation to the student in other ways. The teacher or another student can read the problem or problems can be prerecorded on tape.

The vast majority of students who cannot solve story problems are not restricted by their ability to read, but by their limited understanding of mathematics. They are frequently proficient in computation, but are unable to identify which operation should be used in a particular context. The more complex the context (such as rate problems, problems which require more than one operation, or problems with extraneous information) the more difficulty students have. It would not be difficult to determine whether a student's problem is due to reading difficulties or to a limited understanding of mathematics.

The vast majority of students who cannot solve story problems are not restricted by their ability to read, but by their limited understanding of mathematics.

Whatever the source of the student's difficulty, the most inappropriate response is to use "low verbal" approaches to instruction and the statement of problems. Some textbook publishers have taken this approach. Not only has this approach been shown to be ineffective, it denies students the opportunity to read in mathematics.

Teachers should make regular reading assignments in mathematics. Typically teachers rely on classroom instruction to communicate information, and rely on the textbook to provide exercises for practice. Students may use the expository sections of the text to look at the sample solutions to exercises, but they seldom read the text itself. Teachers in elementary grades should periodically teach reading using material in the mathematics text, and reinforce this instruction with regular reading assignments, followed by questions on the reading. Teachers in secondary schools should also follow reading assignments with questions on the reading. Where severe reading problems exist, the reading specialist should be consulted.

Mathematics has a very technical vocabulary which increases the difficulty of reading mathematics material.

Mathematics has a very technical vocabulary which increases the difficulty of reading mathematics material. This difficulty is compounded by the use of very common words such as "similar" in very restricted ways. The premature introduction of technical vocabulary may inhibit rather than enhance understanding. Generally it is better to develop understanding of a concept and then apply the proper vocabulary once the concept is understood. Beginning with a definition may focus too much attention on rote recall of the definition itself. The students may never progress to a thorough understanding of the concept being defined.

The Directed Reading Thinking Activity

The Directed Reading Thinking Activity (DRTA) is a general model for teaching subject-area reading that incorporates much of the appropriate knowledge and recent research regarding reading comprehension instruction. This is described in detail in *A Guide to Curriculum Planning in Reading*, which should be consulted by those wishing a more complete description of the activity. The DRTA provides an instructional framework for integrating specific reading strategies or processes into the study of mathematics while encouraging development of critical reading and thinking that can be applied to all subject areas.

The DRTA involves five stages: preparation for reading, setting a purpose, guided silent reading, discussion, rereading plus reflection and extension activities. The following sections will describe each of these, offering suggestions for instruction and application by teachers of mathematics.

Preparation for Reading

Before giving a reading assignment, examine the material to determine features that would help student comprehension and to identify both unfamiliar vocabulary and concepts that might involve difficulties for the students. Next, decide whether students have the necessary concepts and vocabulary knowledge to achieve a satisfactory level of meaning from the text. Vocabulary instruction research seems to indicate that a focus on concepts and meanings in familiar contexts is more profitable than a rote learning of new words in isolation. Thus one should teach unfamiliar vocabulary essential to understanding text material in rich contextual settings that are relevant and interesting before and during reading.

Setting a Purpose

The more thoroughly teachers familiarize students with the organization of the text, the more likely students are to achieve comprehension. Before reading a selection, have students preview it, examining the arrangement of material and such features as headings, italicized words, summary statements, and so on to develop predictions concerning content. In mathematics, reading material is frequently accompanied by illustrations or examples with references in the text to the illustration or the example. Students who follow only the text will not get the full benefit of the descriptive material, just as students who look only at the illustration or example will frequently miss valuable instruction provided by the description. Through the preview, students begin to identify a purpose for their reading. The teacher should focus students' attention on important concepts contained in the text.

. . . students who look only at the illustration or example will frequently miss valuable instruction provided by the description.

Guided Silent Reading

Once reading purposes are clearly in mind, students should read the assigned material silently at their individual pace.

Discussion, Rereading, and Reflection

Following purposeful silent reading, students should be guided in the discussion of the reading content. Provide them an opportunity to talk about the content in relation to their purposes for reading it. They should discuss whether the information acquired was sufficient to answer their questions and fulfill expectations. Where relevant, they should describe how and why purposes for reading changed as they completed the assignment.

Critical Thinking

During discussion, ask questions requiring students to go beyond the specific details and think critically about the overall concepts and longer messages in the text. Have them verify their reasons by rereading sections to support interpretations or to identify inconsistencies in the author's reasoning. Rereading passages can be done either orally or silently, but should always have a definite purpose.

Extension Activities

Extension activities serve to help students expand upon information gained from the reading. Such activities, therefore, provide students opportunities to incorporate new ideas and information into their background understanding.

Additional information on the Directed Reading Thinking Activity is available in *A Guide to Curriculum Planning in Reading*. Readers may also wish to consult *Reading in the Content Area of Mathematics* (Nolan, 1984).

Environmental Education and Mathematics

The Department of Public Instruction curriculum guide for environmental education, *A Guide to Curriculum Planning in Environmental Education*, advises that environmental education be infused into the instruction of other disciplines rather than be taught as a separate course. Many applications of mathematics can be found in the environment. Some examples are provided in this section.

Many applications of mathematics can be found in the environment.

Measurement

One of the most frequently applied topics from mathematics is measurement. It is natural to have children study the environment by measuring. Measures associated with weather can be conducted at all grade levels. These include temperature, precipitation, humidity, and wind velocity. Measures of length, area, and volume can be applied to natural objects such as trees. Indirect measure can be used to find the diameter or height of a tree. Estimates can be made of the measures of water in a lake or the width of rivers.

Statistics

Statistical procedures are frequently used in studies of the environment. Wild animal populations and migrations may be studied by tagging a sample from a population and then doing subsequent studies in which representatives from the sample are rediscovered. Statistical studies are frequently used to identify endangered species before that species becomes extinct. Statistical studies can also be used to project population growth and estimate the capacity of an environment to support a population of a particular size. Graphs are frequently used to communicate statistical information about the environment.

Geometry

Geometry can be used to describe an environment. Objects in nature can be described according to the degree to which they approximate geometric shapes. Measurement properties of geometric shapes may be used to estimate measures of objects in nature. For example, a kettle may resemble a hemisphere, so knowledge of properties of hemispheres may be applied to the description of these depressions. The concept of slope may be applied to descriptions of surfaces, and techniques for determining the slope of a surface can be explored. The Pythagorean Theorem has many applications in indirect measurement.

Arithmetic and Algebra

Arithmetic and algebra can be used in quantitative studies of the environment. The erosion caused by a stream can be estimated in inches per year and that estimate used to project the amount of soil lost in five years, fifty years, or five hundred years. The amount of energy production necessary for a community can be calculated based on a historically determined average rate of consumption per person.

These are just examples and are not intended to provide a comprehensive list of applications of mathematics in environmental studies. In teaching mathematics, however, teachers are encouraged to use applications both to motivate the study of each topic and to illustrate its usefulness to the students. Studies of the environment provide a rich source of opportunities to convince students that mathematics can be both fun and useful. At the same time it meets the goal of integrating environmental education into at least one other school discipline.

Readers are encouraged to consult *A Guide to Curriculum Planning in Environmental Education* for additional information.

Career Education and Mathematics

Mathematics is known as the "critical filter" because it restricts choices in education and careers for those whose knowledge and understanding of mathematics is limited.

Mathematics is known as the "critical filter" because it restricts choices in education and careers for those whose knowledge and understanding of mathematics is limited.

Few students in elementary and secondary schools become mathematicians. Most students will spend some time in occupations which require some knowledge of mathematics. The curriculum defined in sections 2 through 8, along with the informal mathematics program described in section 9 and Appendix K, are intended to provide the necessary mathematics preparation for most non-technical careers. While it is possible to hold some entry-level positions with less mathematics, students should have a curriculum sufficiently rich to prepare them for a variety of careers beyond entry-level positions. Furthermore, since some students make a belated decision to attend a college or another post-secondary institution, a curriculum rich enough to allow both options is advised.

The College Board, in its 1983 report *Academic Preparation for College*, identified Basic Academic Competencies as "the broad intellectual skills essential to effective work in all fields of college study" (College Board, 1983). Basic Academic Competencies are listed for reading, writing, speaking and listening, mathematics, reasoning, and studying.

Students who enter the work force directly from high school need many of the same academic competencies as do those going on to college.

In a subsequent report, *Academic Preparation for the World of Work* (College Board, 1984), the College Board observed that "Academic preparation is not just for the college bound. Students who enter the work force directly from high school need many of the same academic competencies as do those going on to college." This conclusion was reached as a result of discussions with 200 business leaders and educators. The Basic Academic Competencies, according to business people "are as necessary for success in the world of work as they are for success in college. The message to schools is clear: work-bound students, whether following vocational or general education courses of study, shouldn't be taken off the academic track."

A number of resources are available to teachers interested in informing students of the importance of mathematics in careers. The Mathematical Association of America has developed a videotape series, *Mathematics at Work in Society: Opening Career Doors*. Those tapes are designed to increase awareness of the connection between mathematics and careers for students of grades 8 and above. They are available on loan from: Mathematics Supervisor, Wisconsin Department of Public Instruction, P.O. Box 7841, Madison, Wisconsin 53707.

Another videotape series, *Multiplying Options and Subtracting Bias*, focuses on the impact of the critical filter on young women. More often than males, females restrict their career options by electing not to take more mathematics than is required. These tapes are available in four versions. One is for students, one for teacher, one for counselors, and one for parents. They are also available on loan from the DPI mathematics supervisor at the above address.

The ERIC Clearinghouse on Adult, Career, and Vocational Education provides information on career education projects in schools, including a report, *Ten Mathematics Projects and Career Education Infusion* (Taylor, 1979). Further information can be obtained by writing: The National Center for Research in Vocational Education, Ohio State University, 1960 Kenny Road, Columbus, Ohio 43210. Projects can also be identified using a regular ERIC search on mathematics and careers.

Computers and Mathematics

Many references have been made throughout this guide to the increasing role of computers in mathematics. This is not limited to the use of computers to teach mathematics or the use of computers to do mathematics which was formerly done by other methods. Computers are making some traditional mathematics obsolete. At the same time they are increasing the importance of mathematics topics which were not considered fundamental. Symbol manipulation in algebra is an example of a traditional topic rendered much less fundamental. Topics in discrete mathematics are becoming much more fundamental. Curriculum committees are urged to study the reports which address this issue (Corbitt, 1984; Fey, 1984; Hansen, 1984) and begin now to plan an evolving curriculum which accommodates this increasing role of computers in mathematics.

As an instructional tool, the computer also adds a new dimension to the mathematics program. Students at all grade levels can use Logo to acquire geometric insights which are very difficult to achieve otherwise. Software which develops estimation skills is a very valuable component of the mathematics program. Simulations offer a variety of experiences, including pattern searches. Rather than define the curriculum exclusively in terms of traditional textbook-oriented instruction, curriculum committees are urged to consider the advantages of computer-related mathematical experiences.

While mathematics and computers have an integrated role to play in school instructional programs, curriculum committees should be aware of the distinctions between mathematics and computer science, and should not allow computer instruction which is not directly related to the mathematics curriculum to reduce the amount of time allocated to mathematics instruction.

. . . curriculum committees . . . should not allow computer instruction which is not directly related to the mathematics curriculum to reduce the amount of time allocated to mathematics instruction.

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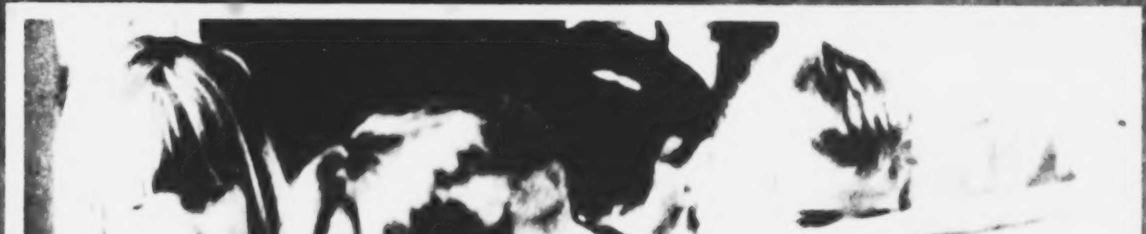
Resources

12

[illegible] Organization



The National Council of Teachers of Mathematics

The National Council of Teachers of Mathematics (NCTM) has for its objective the advancement of mathematics teaching in elementary schools, junior and senior high schools, two-year colleges, and teacher-education colleges. All persons interested in mathematics or the teaching of mathematics are eligible for membership.

Brochures describing NCTM membership, publication, and convention services are available free on request from the Headquarters Office, 1906 Association Drive, Reston, Virginia 22091 (703/620-9840). The Source: STJ228; CompuServe: 75445, 1161.

The Wisconsin Mathematics Council

The Wisconsin affiliate of the National Council of Teachers of Mathematics is the Wisconsin Mathematics Council. Like NCTM, the Wisconsin Mathematics Council promotes the advancement of mathematics teaching in elementary schools, junior and senior high schools, two-year colleges, and teacher-education colleges. All persons interested in mathematics or the teaching of mathematics are eligible for membership.

The Wisconsin Mathematics Council regularly publishes a newsletter and a journal, *The Wisconsin Teacher of Mathematics*. The Council holds its annual conference at the Green Lake Center each year on the first Friday and Saturday in May. It regularly sponsors two fall conferences in October and a variety of special one-day conferences during the year.

For further information on membership or services, write to Wisconsin Mathematics Council, P.O. Box 10127, Milwaukee, Wisconsin 53210 or call (608) 266-7712.

Milwaukee Area Mathematics Council

The Milwaukee Area Mathematics Council is also an affiliate of the National Council of Teachers of Mathematics. The Milwaukee Area Mathematics Council holds meetings in the Milwaukee area about three times per year. Meetings may involve dinner with an after-dinner speaker or a visit to a local business, industry, or educational institution with a program of interest to the members.

The Milwaukee Area Mathematics Council has no permanent mailing address. For further information contact Mathematics Supervisor, Wisconsin Department of Public Instruction, P.O. Box 7841, Madison, Wisconsin 53707.

The Milwaukee Educational Computing Association

Persons interested in computers and their role in elementary and secondary education are invited to consider membership in the Milwaukee Educational Computing Association. Regular meetings are held on Saturday mornings for presentations and discussions of topics of interest to members. For further information contact Henry S. Kepner, Jr., Department of Curriculum and Instruction, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201.

The Wisconsin Department of Public Instruction

The Wisconsin Department of Public Instruction offers programs and personnel which can assist local districts in planning or implementing aspects of their mathematics programs.

The Department has two mathematics consultants. Persons seeking assistance with any or all aspects of their mathematics programs are invited to call Donald Chambers at (608) 266-7712 or James Moser at (608) 267-9170, or to write to either in care of Wisconsin Department of Public Instruction, 125 South Webster Street, P.O. Box 7841, Madison, Wisconsin 53707. Mathematics-related publications are available from the Publications Office, Wisconsin Department of Public Instruction.

The Department's consultants may provide assistance with the following mathematics program components.

- Curriculum development
- Textbook selection
- Program evaluation
- Sex equity
- Competency testing
- Chapter I programs
- Gifted and talented programs

Consultants may also be requested to participate as part of teacher inservice activities. These programs may include (but are not necessarily limited to) the following.

- Effective instruction
- Current trends in mathematics education
- Teacher certification issues

Consultants can also answer questions related to a district's participation in professional development activities under the Education For Economic Security Act. An extensive collection of audiotapes and videotapes are available on loan for use in local inservice or professional development activities. A catalogue is provided in the next section, Media Resources.

The mathematics consultants can also provide liaison with other Department of Public Instruction programs and personnel including the following.

- Microcomputers and instructional technology
- Education Consolidation and Improvement (ECIA) 1981 Block Grants
- Vocational education
- Education of the handicapped

Media Resources

The Wisconsin Department of Public Instruction has audiotapes and videotapes which may be useful for teacher inservice programs or for individual professional development. A catalogue of available programs follows. Included among the audiotapes are major presentations from the 1980 Annual Meeting of the National Council of Teachers of Mathematics; the 1980 Fourth International Congress on Mathematical Education (note that some speakers at the Congress used languages other than English); the 1983 Annual Meeting of the National Council of Teachers of Mathematics; and the 1985 Green Lake Meeting of the Wisconsin Mathematics Council. Videotapes for creating awareness of problems of mathematics anxiety and sex-role stereotyping are available, as well as mathematics for careers, and a framework for teaching mathematics effectively.

All the indicated materials are available on loan from the Wisconsin Department of Public Instruction. One or two programs at a time may be borrowed for two weeks. The most efficient way to arrange for a loan of program tapes is to call 608/267-9279 to indicate the programs desired and the preferred dates.

Audiotapes

1980 Annual Meeting of the National Council of Teachers of Mathematics

Tape Nos.

1 & 1A **Agenda for a Decade of Action.** Shirley A. Hill, President, National Council of Teachers of Mathematics, University of Missouri, Kansas City, MO. NCTM releases its set of major policy recommendations for the mathematics curriculum of the 1980s. This is the beginning step in a concerted effort toward positive change throughout the decade.

17 **Effective Numeracy in a Complex World: A Challenge for Elementary Schooling.** Max Bell, University of Chicago, Chicago, IL. Numeracy needs are both substantially and qualitatively different from even a decade ago. Our present responses are demonstrably inadequate. Some first steps toward more adequate responses will be prescribed.

40 **Back to the REAL Basics.** Ann P. Kahn, Fairfax County School Board, Fairfax, VA. How can math teachers have an impact on critical educational issues? How are teachers of mathematics affected by parental concerns regarding teaching and testing in the 1980s?

78 **Mathematics Offerings in our Schools.** James Fey, University of Maryland, College Park, MD. Recommendation 6 urges that more mathematics be required of elementary and secondary students and that a greater range of options be available. Deceptively simple and attractive as these proposals are, they will not go unchallenged.

114 **Imminent Instructional Implications.** Shirley M. Frye, Scottsdale School District, Phoenix, AZ. The important implications of recommendations 2, 4, and 6 will be related to the elementary school teacher who is a vital key in the educational life of the students of the 1980s.

137 **What Mathematics Will We Teach in the 1980s? An NCTM PRISM Project Report for Elementary Schools.** Marilyn Suydam and Peggy Kasten, Ohio State University, Columbus, OH. Planning changes in what and how you teach? This session allows you to compare your ideas with responses of others questioned about curricular concerns for the 1980s.

174 **Promoting Problem Solving: A Plan for the Present.** Carole Greenes, Boston University, Boston, MA. In the 1980s, mathematics education will focus on problem solving. Until curriculum is developed for this purpose, no more than simple modification of existing materials and new teaching strategies is required.

206 **Base the Curriculum on Problem Solving? How? Why?** Jeremy Kilpatrick, University of Georgia, Athens, GA. The goal of a school mathematics curriculum organized around problem solving will not be attained in the 1980s. Is such a goal attainable? Is it desirable?

Tape Nos.

245 **What Mathematics Will We Teach in the 1980s? An NCTM PRISM Project Report for Secondary Schools.** John Higgins and Peggy Kasten, The Ohio State University, Columbus, OH. Planning changes in what and how you teach? This session allows you to compare your ideas with responses of others questioned about curricular concerns for the 1980s.

269 **An Appraisal of Substance and Process in School Programs.** John I. Goodlad, University of California-Los Angeles, Los Angeles, CA. There are vast differences in the knowledge gained by and a lack of equity in the modes of learning to which young people are being exposed. Reconstruction of the curriculum to provide balance and efforts to provide access to the "common school" must rank high on any agenda of school reconstruction during the 1980s.

306 **The Middle School: A Time to Stop and a Time to Start.** Joan E. Worth, University of Alberta, Edmonton, Alberta. The recommendations for the 1980s are most exciting, most demanding, and most important at the middle school level. Why? How can we meet this challenge?

340 **Technology and Education—the Impact of Computing Technology on the Objectives of Elementary and Secondary Education, and Beyond.** Donald Kreider, Dartmouth College, Hanover, NH. Curricula at all levels must be examined and modified to exploit the power of the new computing technology and to prepare our students for its effective and tasteful use.

361 **Mathematics and Rodney Dangerfield: Changing the Image.** Jack Price, Vista Unified School District, Vista, CA. The image of mathematics can be changed in the 1980s by improving the professionalism of mathematics teachers, supervisors, and teacher educators and by enlisting and retaining public support.

399 **Bettering the Basics: Pruning Past Practices and Insightfully Implementing Innovation.** Albert P. Shulte, Oakland Schools, Pontiac, MI. What topics should be added to, or deleted from, the mathematics curriculum in the 1980s? What teaching styles or strategies should be used?

437 **Effecting Change: What is Necessary? An NCTM PRISM Project Report for Inservice and Pre-service Educators.** Alan Osborne and Peggy Kasten, Ohio State University, Columbus, OH. The responses of different groups, lay and professional, indicate that changing the curriculum for the 1980s will require varied approaches and efforts. What are the implications for leadership?

458 **The Impact of Media on Thinking.** Neil Postman, New York University, New York, NY.

474 **PRISM: Canada Project.** George W. Catheart, Sharon Forth, Thomas E. Kieren, Joan E. Worth and Walter H. Worth, University of Alberta, Edmonton, Alberta. The panel will discuss aspects of the cross-Canada PRISM study. This will include comparisons among responding groups, discussion of provincial perspectives, and comparisons with United States results.

Tape Nos.

495 **Effective Instruction and Productive Learning.** Paul R. Trafton, National College of Education, Evanston, IL. Students need sound, sensitive instruction if they are to learn mathematics successfully. We will examine some components of effective instruction with an emphasis on their link to productive learning.

525 **The Long Road Ahead.** Max A. Sobel, President-elect, National Council of Teachers of Mathematics, Montclair State College, Upper Montclair, NJ. A lot has been said! But what will be done? Let us begin our campaign now to teach mathematics better in the '80s...and the '90s...and thereafter.

Fourth International Congress on Mathematical Education, 1980

Tape Nos.

081 **Major Problems of Mathematics Education.** Hans Freudenthal, Institut Ontwikkeling Wiskunde Onderwijs, The Netherlands.

114 & 124 & 134 **Successes and Failures of Mathematics Curricula in the Past Two Decades.** H. Brian Griffiths, University of Southampton, Southampton, England (Organizer); Panel: Francois Colmez, IREM, Universite de Paris, Paris, France: Ubiratan D'Ambrosio, IMECC-UNICAMP, Campinas, Sao Paolo, Brazil; Peter Damerow, Max-Planck-Institut fur Bildungsforschung, Berlin, Germany; Stephen S. Willoughby, New York University, New York, USA. The speaker will deal with the general problem of what is meant by success and failure of mathematics curricula when these are viewed in the social contexts in which they are applied. The panel members will then discuss examples from their own societies.

118 **Critical Variables in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Richard E. Snow, Stanford University, Stanford, California, USA. The use of critical variables to organize research.

131 **Is Calculus Essential?** Panel: Margaret E. Rayner, St. Hilda's College, Oxford, England (Organizer); Fred Roberts, Rutgers Unviersity, New Brunswick, New Jersey, USA. Should calculus keep its unique position as the core of post-secondary and undergraduate mathematics? If not calculus, what should replace it?

151 **Young Children's Acquisition of Language and Understanding of Mathematics.** Hermine Sinclair, University of Geneva, Geneva, Switzerland.

Tape Nos.	
163 & 173	**Curriculum Recommendations for the 1980s by several National Committees.** Panel: Mohammed El Tom, University of Khartoum, Sudan, *Problems of Curriculum Development in Sudan*; W. H. Cockcroft, The New University of Ulster, Londonderry, Northern Ireland, *The Effect of Government Enquiries into the Teaching of Mathematics*; Shirley Hill, University of Missouri, Kansas City, Missouri, USA, *Policy Recommendations of the NCTM for the Mathematics Curriculum of the 1980s*; David F. Robitaille, The University of British Columbia, Vancouver, Canada (Organizer), *Trends in Mathematics Education in Canada and Projections into the 1980s*. Each of the four panelists will report on current developments in the mathematics curriculum in his/her country. Trends and issues will be identified.
16B & 27A & 43B	**Roots of Failure in Primary School Arithmetic** Panel: Guy Brousseau, Universite de Bordeaux I, Talence, France; Luis Dante, Universidad Paulista Estadual, Rio Claro, Sao Paolo, Brazil; Alistair McIntosh, Principal Mathematics Advisor, Leicester, England; Mumba Mwenda, Ministry of Education, Lusaka, Zambia; Frederique Papy, Brussels, Belgium; Hassler Whitney, Institute for Advanced Study, Princeton, New Jersey, USA (Organizer). When children fail to learn arithmetic in primary school, what are the roots of this failure? What approaches can be taken towards a cure?
179	**Some Critical Variables Revisited (Begle Memorial Series on Research in Mathematics Education).** Christine Keitel-Kreidt, Institute fur Didaktik der Mathematik, Bielefeld, Federal Republic of Germany; Donald J. Dessart, University of Tennessee, Knoxville, Tennessee, USA. An extension and updating of E. G. Begle's survey of the empirical literature regarding curriculum variables.
21B	**Critical Variables in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Herbert J. Walberg, University of Illinois at Chicago Circle, Chicago, Illinois, USA. Problems of synthesizing research.
229	**Some Critical Variables Revisited (Begle Memorial Series on Research in Mathematics Education).** L. Ray Carry, University of Texas, Austin, Texas, USA: Jens H. Lorenz, Institut fur Didaktik der Mathematik, Bielefeld, Federal Republic of Germany. An extension and updating of E. G. Begle's survey of the empirical literature regarding student variables.
231	**Language and the Teaching of Mathematics.** A. G. Howson, University of Southampton, Southampton, England. Language and mathematics education interact in a variety of ways. In this talk, an attempt will be made to identify particular areas or problems suitable for further research and investigation.
23Ba & 23Bb	**Some New Directions for Research in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Richard E. Mayer, University of California, SantaBarbara, California, USA; Edward A. Silver, San Diego State University, San Diego, California, USA. Research on memory.

Tape Nos.

251 **The Computer as Carrier of Mathematical Culture.** Seymour Papert, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA. "I take from Piaget a model of children as builders of their own intellectual structures. But like other builders, they need building materials and find them in the surrounding culture. Deep change in what structures children develop 'spontaneously' requires change in the culture, something that is not easy for educators to induce. However, the ever-increasing penetration of the computer presence—ideas as well as machines—is beginning to induce cultural changes of a relevant kind. I believe that the biggest challenge for research in mathematics education is to understand these trends and find ways to deflect them in directions that will facilitate the 'spontaneous' development of mathematical thinking. I shall talk about theoretical steps already taken in this direction and about some first experimental results."

272 **Analogies and Metaphors to Explain Godel's Theorem.** Douglas R. Hofstadter, Indiana University, Bloomington, Indiana, USA.

311 **Back-To-Basics: Past, Present, Future.** Max Sobel, Montclair State College, Upper Montclair, New Jersey, USA. The 1970s ending with a "back-to-basics" movement. This session will review the meaning and explore the implications of the trend for the 1980s.

318a & 318b & 41C **The Death of Geometry at the Post-Secondary Level.** Panel: Jean Dieudonne, Paris, France; Branko Grunbaum, University of Washington, Seattle, Washington, USA (Organizer); Robert Osserman, Stanford University, Stanford, California, USA; H. Zeitler, Bayreuth, Federal Republic of Germany. Many consider geometry "dead" or irrelevant to the education and work of contemporary mathematicians. Is it indeed? Can or should anything be done about it?

31Ba & 31Cb **Some New Directions for Research in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Thomas L. Good, University of Missouri, Columbia, Missouri, USA; Richard Lesh, Northwestern University, Evanston, Illinois, USA. Research on teaching.

31Ca & 31Cb **What Should be Dropped from the Secondary School Mathematics Curriculum to Make Room for New Topics?** Panel: Floyd Downs, Hillsdale High School, San Mateo, California, USA; Fou-Lai Lin, National Taiwan Normal University, Taipei, Taiwan; C.P. van Nieuwkastelle, Technische Hogeschool Eindhoven, Eindhoven, Netherlands; Zalman Usiskin, University of Chicago, Chicago, Illinois, USA (Organizer). The time devoted to mathematics cannot always be increased to cover important new mathematical developments and applications. Each panelist will discuss, in terms of his own country's experience, what topics might be deleted.

Tape Nos.	
322 & 332	**Teaching Mathematics in a Second Language.** Panel: Mauricio Gneere, UNICAMP, Campinas, Sao Paolo, Brazil; Rayna Green, Dartmouth College, Hanover, New Hampshire, USA; Mme. Moumouni, University of Niamey, Niamey, Niger; Busa Xaba, University College, Kwaluseni, Swaziland; Althea Young, Ministry of Education, Kingston, Jamaica, West Indies. The panel will discuss practical difficulties of teaching mathematics in a language which is not the usual language of students outside of the classroom. Specific discussions will be focused on Native Americans, on South American villages, on Caribbean creole, and on European immigrants. There will also be reference to possible intrinsic difficulties found in some languages with respect to mathematical concepts and structure.
325 & 528 & 619	**The Child's Concept of Number.** Panel: Claude Comiti, Universite de Grenoble, Saint Martin d'Heres, France; Karen Fuson, Northwestern University, Evanston, Illinois, USA; Tom Kieren, University of Alberta, Edmonton, Alberta, Canada (Organizer); Hugh Philip,Macquarie University, North Ryde, New South Wales, Australia; Shuntaro Satoh, Fukushima University, Fukushima, Japan; Peter Sorger, Padagogische Hochschule Westfalen-Lippe, Munster, Federal Republic of Germany; Gerhard Steiner, Universitat Basel, Basel, Switzerland. This mini-conference will focus on the mechanism and processes children use in building up number concepts. It will include discussions of the development of pre-number concepts through fractional number concepts.
337a & 337b	**Alternative Approaches to Beginning the Teaching of Calculus, and the Effectiveness of These Methods.** Panel: Hugh Neill, University of Durham, Durham, England (Organizer); Georges Papy, Universite Libre de Bruxelles, Bruxelles, Belgium; Daniel Reisz, Universite de Dijon, IREM, Dijon, France.
33Ba & 33Bb	**Some New Directions for Research in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Robert B. Davis, University of Illinois, Urbana, Illinois, USA: Gunnar Gjone, Pedagogisk Seminar, Oslo, Norway; John P. Keeves, Australia Council for Educational Research, Hawthorn, Victoria, Australia. Their implications for mathematics education; a research agenda for 1984.
412 & 422	**Special Assistance for the Beginning Teacher.** Panel: Edith Biggs, London, England; Mervyn Dunkley, Macquarie University, North Ryde, Australia; Gwen Shufelt, University of Missouri, Kansas City, Missouri, USA (Organizer); Alexander Tobin, School District of Philadelphia, Philadelphia, Pennsylvania, USA. Each panelist will present models of successful progress that have provided assistance to the beginning teacher. Procedures and personnel involvement will be discussed. Time will be allowed for questions and discussion by the audience.
418	**Critical Variables in Mathematics Education (Begle Memorial Series on Research in Mathematics Education).** Edwin E. Moise, New York, USA; Lauren B. Resnick, University of Pittsburgh, Pittsburgh, Pennsylvania, USA. The kind of empirical research that would be most useful to mathematics education.

Tape Nos.	
427 & 43Aa & 43Ab	**The Making of a Professional Mathematics Teacher.** Gerald Rising, SUNY at Buffalo, New York, USA; Reactors: Bent Christiansen, Royal Danish School of Educational Studies, Copenhagen, Denmark (Organizer); Geoffry Howson, University of Southampton, Southampton, England; Alfred Vermandel, University of Antwerp, Antwerp, Belgium. Basic aspects of professionalism. What knowledge, abilities, qualities, and attitudes should a teacher acquire? From mathematics? From other fields? During initial education and/or later? Where? When? How?
42B	**Some Critical Variables Revisited (Begle Memorial Series on Research in Mathematics Education).** Stephen I. Brown, State University of New York, Buffalo, New York, USA: Nicholas A. Branca, San Diego State University, San Diego, California, USA. Problem solving.
438a & 438b	**A Procedural Approach Toward Mathematics Education.** Patrick Suppes, Stanford University, Stanford, California, USA; Reactor: John G. Harvey, University of Wisconsin, Madison, Wisconsin, USA.
511	**Some Experiences in Popularizing Mathematical Methods.** Hua Loo-keng, Academia Sinica, Peking, China. An account is given of personal experiences in popularizing mathematical methods among millions of ordinary workers and peasants in China over the past 15 years. The chief principles and requirements in popularization are considered. Illustrations are given for the development and understanding of problem-solving techniques through the application and evaluation of some methods such as diophantine approximation optimization techniques, network analysis, and statistics. The features of social education instead of school education in mathematics are exemplified in the context.
521	**Vigor, Variety and Vision - The Vitality of Statistics and Probability.** I. J. Good, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, USA. A presentation of the case for a greater emphasis on statistics and probability in the school curriculum.
539a & 539b	**The Functioning of Intelligence and the Understanding of Mathematics.** Richard Skemp, University of Warwick, Coventry, England; Reactors: Laurie Buxton, Inner London Education Authority, London, England; Nicholas Herscovics, Concordia University, Montreal, Quebec, Canada. The relation of earlier work on the understanding of mathematics to a new model of intelligence.
553 & 568	**Do We Still Need Fractions in the Elementary Curriculum?** Peter Hilton, Case-Western Reserve University, Cleveland, Ohio, USA (Organizer); Reactors: P. Braunfeld, University of Illinois, Urbana, Illinois, USA; Mary Laycock, Nueva Learning Center, Hillsborough, California, USA; Pearla Nesher, Haifa University, Haifa, Israel.

Tape Nos.

564 & 574 **Participation of the Handicapped in Mathematics.** Panel: Dudley C. Blane, College of St. Paul and St. Mary, Gloucester, England; Robert Dieschbourg, Institute Pedagogique, Walferdange, Luxembourg; Carol Greenes, Boston University, Boston, Massachusetts, USA; Esther Pillar Grossi, Grupo de Estudos sobre o Ensina de Mathematica de Porto Allegre, Brazil; Muriel Healey, Macquarie University, North Ryde, New South Wales, Australia; Peter N. Richards, University of Bath, Avon, England. Based on research, the participation of the physically, emotionally, and/or mentally handicapped students in school and post-secondary mathematics can be increased.

614 & 621A **Special Problems of Women in Mathematics.** Panel: Patricia L. Casserly, Educational Testing Service, Princeton, New Jersey, USA: Erika Schildkamp-Kundiger, Universitat Saarbrucken, Saarbrucken, Federal Republic of Germany (Organizer); Brigitte Senechal, IREM de Basse Normandie, Caen, France; Sheila Tobias, Washington School of Psychiatry, Washington, D.C., USA; Paulin Yalo, Universite Nationale du Benin, Cotonou, Benin. Empirical research exploring causes of sex-related achievement differences and career plans will be discussed.

1983 Annual Meeting of the National Council of Teachers of Mathematics

Tape Nos.

58 **Courage to Change.** Stephen S. Willoughby, President, National Council of Teachers of Mathematics, New York University, New York, NY. Education propagates today's knowledge. But progress depends on innovation as well as knowledge. Mathematics teachers need the courage to teach for both knowledge and creativity.

105 **Role of the Teacher in the Computer Era.** Lola J. May, Winnetka Public Schools, Winnetka, IL. Teachers are still needed to do essential stages in teaching such as teach concepts, change method, problem solving, and creating happenings.

213 **A Proposal for Reforming the Secondary School Mathematics Curriculum.** Zalman Usiskin, University of Chicago, Chicago, IL. Technological innovations, mathematical advances, and societal developments have made the present curriculum obsolete. Reform is needed. What principles should guide reform? What result? How to implement?

317 **Can Kids Eat Hamburgers 180 Days a Year?** Earl G. Ockenga, Price Laboratory School, University of Northern Iowa, Cedar Falls, IA. Teaching junior high school is tricky. The results we expect don't always happen. What to do? Try doing the unexpected. It can make all the difference.

536 **Mathematics Teachers Count!** Max A. Sobel, Past President, National Council of Teachers of Mathematics, Montclair State College, Upper Montclair, NJ. As has always been the case, the teacher is the key to the learning process. Let's explore the qualities of outstanding teachers.

1985 Annual Meeting of the Wisconsin Mathematics Council

Tape Nos.

073 **Mathematics and Diamonds.** Zalman Usiskin, University of Chicago. Like diamonds, mathematics is multifaceted. The goal of teaching mathematics is, as in diamond cutting, to make evident as many of its facets as one can from as many directions as possible; thereby improving its luster to the casual observer and its value to the serious appraiser.

080 **Techniques For Teaching Problem Solving.** Joan Duea, Price Laboratory School, Cedar Falls, Iowa. Do you have the techniques and materials necessary to teach problem solving? Do your students have a positive attitude toward solving problems? Come and collect ideas on ways to make problem solving an effective learning experience.

090 **Mathematics Curricular Reform and Those Wonderful Computing Machines.** John Moyer, Marquette University. Microcomputers have stimulated loud calls for accelerating the rate of secondary mathematics curricular change. Is this modern-day tortoise about to exchange its shell for a space suit?

100 **Attitudes: Are They Important?.** Elizabeth Fennema, University of Wisconsin-Madison. What are attitudes? What is their relationship to mathematics learning? What causes them? How can they be measured?

110 **Improving K-6 Mathematics Programs and Instruction TODAY.** J. Fred Weaver, University of Wisconsin-Madison. Emphasis will be given to one particular way in which we can work within the context of existing programs to improve instruction along lines that will be consistent with any long-range curriculum goals.

120 **A Critical Response to the New Mathematics Curriculum Guidelines.** John Lucas, University of Wisconsin-Oshkosh; Allan Christenson, Milwaukee Area Technical College; Philip Skoglund, Racine School District; Donald L. Chambers, Wisconsin Department of Public Instruction. Representatives from different segments of the mathematics education community identify the strengths and weaknesses of the new guidelines from their respective points of view.

123 **Wisconsin's New Mathematics Curriculum Guidelines.** Vincent F. O'Connor, Milwaukee Public Schools; Bonnie David, Newbold Elementary School, Rhinelander; Thomas P. Carpenter, University of Wisconsin-Madison; Gail Burrill, Whitnall High School, Greenfield. A presentation of highlights from the new mathematics curriculum guide by a panel of members of the Mathematics Curriculum Task Force.

Videotapes

Multiplying Options and Subtracting Bias. Four videotapes funded by a grant from the U.S. Office of Education and developed at the University of Wisconsin-Madison under the direction of Professor Elizabeth Fennema, were developed to provide information about stereotyping in mathematics, the usefulness of mathematics, the importance of the study of mathematics, and career options. Four half-hour videotapes are available. Each covers the same material but is designed for a different audience. The intended audiences are high school students, high school mathematics teachers, high school counselors, and parents of high school students. The programs are designed to motivate high school females (and males) to continue their study of mathematics. They are also intended to assist teachers, counselors, and parents in encouraging females as well as males to study mathematics.

These videotapes are available on 3/4 inch cassettes.

Mathematics at Work in Society: Opening Career Doors. These four videotapes were developed by the Mathematical Association of America under a grant from the National Science Foundation. The materials were designed to increase awareness of the connection between mathematics and careers for students in grades 8 through 12. Programs include: "An Actuary-What's That?," "Mathematics In Space," "Mathematics: The Language of Research," and "Mathematics: Where Will I Ever Use It?".

These videotapes are available on 3/4 inch cassettes.

Teaching Mathematics Effectively. Based on their research findings, Professors Thomas L. Good and Douglas A. Grouws of the University of Missouri-Columbia have developed a set of key instructional behaviors for the teaching of mathematics. Their model, explained and demonstrated in the videotape, is useful for teaching basic mathematics skills, mathematical concepts, and verbal problem-solving skills to both elementary and secondary students.

This videotape is available on 3/4 inch cassette.

Appendixe[illegible] 13

A. Overview and Analysis of School Math[illegible] K-12
B. Position Paper on [illegible] Skills
C. An Agenda for Action: [illegible]
School [illegible]
D. The Mathematical [illegible] 12:
What Is Still Fundamental [illegible] Not
E. Improving [illegible]
F. School Mathematics [illegible]
G. The Impact of Computing [illegible]
H. [illegible]
I. Recommendations for the Prepar[illegible]
J. Essential Objectives for [illegible]
K. A Three Year Curriculum for [illegible]

Overview and Analysis of School Mathematics Grades K-12*

Conference Board of Mathematical Sciences
National Advisory Committee on Mathematical Education

Curriculum Content

Curriculum content, subject to the flux of accelerating change in all areas of our society, cannot be viewed as a fixed set of goals or ideas; it must be allowed to emerge, ever changing, responsive to the human and technological lessons of the past, concerns of the present, and hopes for the future. With this in mind, no definitive curriculum can ever be recommended. At benchmark 1975 the National Advisory Committee on Mathematics Education (NACOME) sees the following recommendations as reasonable and essential features of a contemporary mathematics curriculum.

- that logical structure be maintained as a framework for the study of mathematics.
- that concrete experiences be an integral part of the acquisition of abstract ideas.
- that the opportunity be provided for students to apply mathematics in as wide a realm as possible—in the social and natural sciences, in consumer and career related areas, as well as in any real life problems that can be subjected to mathematical analysis.
- that familiarity with symbols, their uses, their formalities, their limitations be developed and fostered in an appropriately proportioned manner.
- that beginning no later than the end of the eighth grade, a calculator should be available for each mathematics student during each mathematics class. Each student should be permitted to use the calculator during all of his or her mathematical work including tests.
- that the recommendations of the Conference Board of the Mathematical Sciences 1972 committee regarding computers in secondary school curricula be implemented. NACOME especially underlines recommendations:
  - —that all students, not only able students, be afforded the opportunity to participate in computer science courses.
  - —that school use of computers be exploited beyond the role of computer assisted instruction or computer management systems.
  - —that "computer literacy" courses involve student "hands-on" experiences using computers.
- that all school systems give serious attention to implementation of the metric system in measurement instruction and that they re-examine the current instruction sequences in fractions and decimals to fit the new priorities.
- that instructional units dealing with statistical ideas be fitted throughout the elementary and secondary school curriculum.

**From the Conference Board of the Mathematical Sciences National Advisory Committee on Mathematical Education,* Overview and Analysis of School Mathematics Grades K-12, *1975. Reprinted with permission.*

Position Paper on Basic Mathematical Skills

National Council of Supervisors of Mathematics

Introduction

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

- Declining scores on standardized achievement tests and college entrance examinations;
- Reactions to the results of the National Assessment of Educational Progress;
- Rising costs of education and increasing demands for accountability;
- Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
- Increased awareness of the need to provide remedial and compensatory programs;
- The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

Rationale For the Expanded Definition

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

Defining Basic Skills

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

Ten Basic Skill Areas

Problem Solving. Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations. The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results. Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation. Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills. Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry. Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement. As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

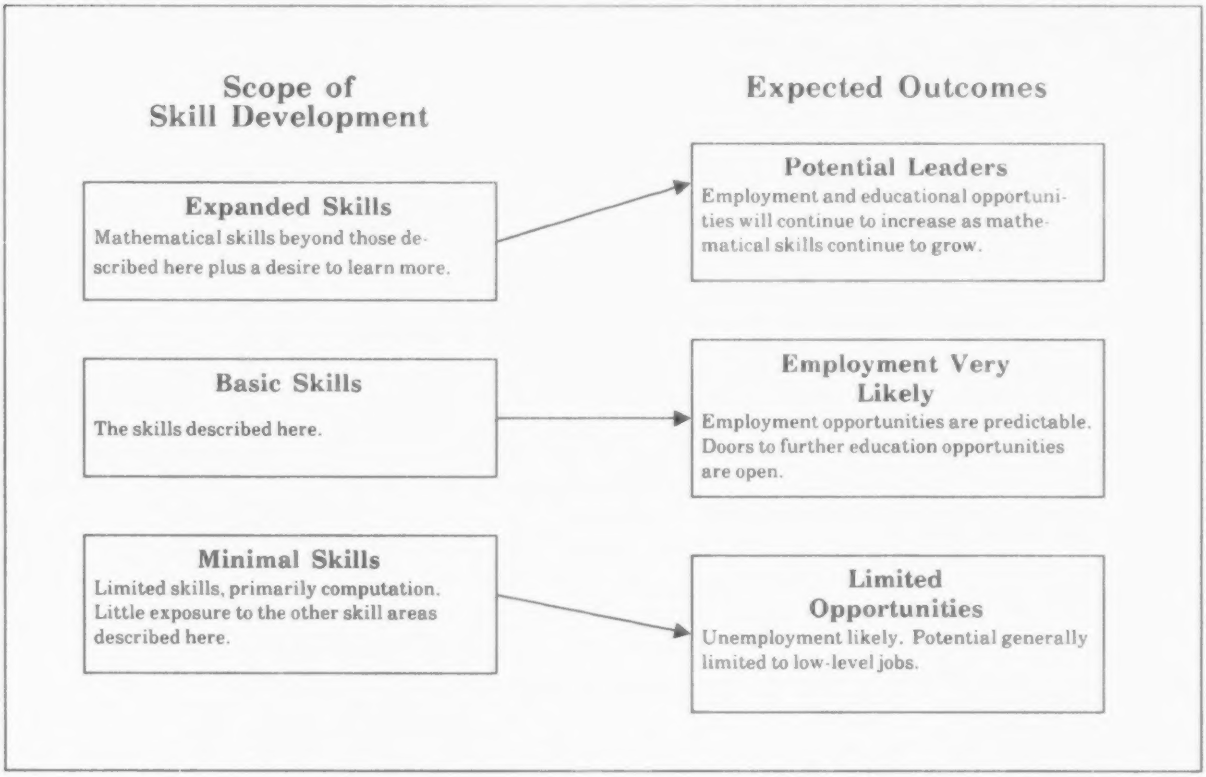
Reading, Interpreting, and Constructing Tables, Charts, and Graphs. Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict. Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy. It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

Basic Skills and the Student's Future

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.



Expanded Skills
Mathematical skills beyond those described here plus a desire to learn more.

Potential Leaders
Employment and educational opportunities will continue to increase as mathematical skills continue to grow.

Basic Skills
The skills described here.

Employment Very Likely
Employment opportunities are predictable. Doors to further education opportunities are open.

Minimal Skills
Limited skills, primarily computation. Little exposure to the other skill areas described here.

Limited Opportunities
Unemployment likely. Potential generally limited to low-level jobs.

Minimum Essentials for High School Graduation

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

Developing the Basic Skills

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations, and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

Evaluating and Reporting Student Progress

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

- Items are not necessarily generated to measure a specific objective or instructional aim.
- The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

Conclusion

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but *forward* to the essential needs of adults in the present and future.

An Agenda for Action: Recommendations for School Mathematics of the 1980s

National Council of Teachers of Mathematics

Preface

In the 1960s there was considerable ferment in mathematics curriculum and instruction. Although public attention was focused on the more visible attempts at program revision, we are aware two decades later that change was more apparent than real.

In the 1970s the concern of the public was directed toward problems evidenced almost exclusively in test scores. Schools have responded to this concern in a variety of ways, but a clear-cut and carefully reasoned sense of direction that looks toward the future has been lacking.

We recognize as valid and legitimate the role of public opinion in the determination of educational goals. But this philosophy is predicated on a well-informed public. Thus, the National Council of Teachers of Mathematics, as an organization of professional educators, has a special obligation to present its responsible and knowledgeable viewpoint of the directions mathematics programs should be taking in the 1980s.

These recommendations represent both realism and responsibility. They are realistic in their attention to hard data. We are fortunate to have more information about mathematics classroom practice than we ever had. This useful information comes principally from a series of studies funded by the National Science Foundation and from two mathematics assessments of the National Assessment of Educational Progress.

The recommendations are responsible to the profession and to the public since they represent a very broad base of belief about objectives and priorities. The Council, funded by the National Science Foundation, conducted an extensive survey of the opinions of many sectors of society, both lay and professional. The project was called Priorities in School Mathematics (PRISM).

Such opinion surveys do not in themselves generate recommendations, but a professional organization, if it is to be responsible, must give them serious consideration as it develops its best-considered advice to society concerning future directions for educational programs.

These recommendations are not the end of our efforts but a beginning. They represent an agenda for a decade of action, and we call on all interested persons and groups to join us in a massive cooperative effort toward better mathematics education for all our youth.

The National Council of Teachers of Mathematics recommends that:

1. problem solving be the focus of school mathematics in the 1980s;
2. basic skills in mathematics be defined to encompass more than computational facility;
3. mathematics programs take full advantage of the power of calculators and computers at all grade levels;
4. stringent standards of both effectiveness and efficiency be applied to the teaching of mathematics;
5. the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;
6. more mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population;
7. mathematics teachers demand of themselves and their colleagues a high level of professionalism;
8. public support for mathematics instruction be raised to a level commensurate with the importance of mathematical understanding to individuals and society.

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Recommendation 1: Problem solving must be the focus of school mathematics in the 1980s

The development of problem-solving ability should direct the efforts of mathematics educators through the next decade. Performance in problem solving will measure the effectiveness of our personal and national possession of mathematical competence.

Problem solving encompasses a multitude of routine and commonplace as well as nonroutine functions considered to be essential to the day-to-day living of every citizen. But it must also prepare individuals to deal with the special problems they will face in their individual careers.

Problem solving involves applying mathematics to the real world, serving the theory and practice of current and emerging sciences, and resolving issues that extend the frontiers of the mathematical sciences themselves.

This recommendation should not be interpreted to mean that the mathematics to be taught is solely a function of the particular mathematics needed at a given time to solve a given problem. Structural unity and the interrelationships of the whole should not be sacrificed.

True problem-solving power requires a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them and the fundamental principles that unify them. Each problem cannot be treated as an isolated example. This recommendation looks toward the need to solve problems in an uncertain future as well as here and now. As such, mathematics needs to be taught as mathematics, not as an adjunct to its fields of application. This demands a continuing attention to its internal cohesiveness and organizing principles as well as to its users.

Recommended Actions

1.1 *The mathematics curriculum should be organized around problem solving.*

- The current organization of the curriculum emphasizes component computational skills apart from their application. These skills are necessary tools but should not determine the scope and sequence of the curriculum. The need of the student to deal with the personal, professional, and daily experiences of life requires a curriculum that emphasizes the selection and use of these skills in unexpected, unplanned settings.

- Mathematics programs of the 1980s must be designed to equip students with the mathematical methods that support the full range of problem solving, including
  - —the traditional concepts and techniques of computation and applications of mathematics to solve real-world problems, the rational and real number systems, the notion of function, the use of mathematical symbolism to describe real-world relationships, the use of deductive and inductive reasoning to draw conclusions about such relationships, and the geometrical notions so useful in representing them;
  - —methods of gathering, organizing, and interpreting information, drawing and testing inferences from data, and communicating results;
  - —the use of the problem-solving capacities of computers to extend traditional problem-solving approaches and to implement new strategies of interaction and simulation;
  - —the use of imagery, visualization, and spatial concepts.

- Mathematics programs should give students experience in the application of mathematics, in selecting and matching strategies to the situation at hand. Students must learn to
  - —formulate key questions;
  - —analyze and conceptualize problems;
  - —define the problem and the goal;
  - —discover patterns and similarities;
  - —seek out appropriate data;
  - —experiment;
  - —transfer skills and strategies to new situations;
  - —draw on background knowledge to apply mathematics.

- Fundamental to the development of problem-solving ability is an open mind, an attitude of curiosity and exploration, the willingness to probe, to try, to make intelligent guesses.

- The curriculum should maintain a balance between attention to the applications of mathematics and to fundamental concepts.

1.2 *The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, processes, and modes of presentation that encompass the full potential of mathematical applications.*

- Computational activities in isolation from a context of application should not be labeled "problem solving."
- The definition of problem solving should not be limited to the conventional "word problem" mode.
- As new technology makes it possible, problems should be presented in more natural settings or in simulations of realistic conditions.
- Educators should give priority to the identification and analysis of specific problem-solving strategies.
- Educators should develop and disseminate examples of "good problems" and strategies that suggest the scope of problem-solving activities for each school level.

1.3 *Mathematics teachers should create classroom environments in which problem solving can flourish.*

- Students should be encouraged to question, experiment, estimate, explore, and suggest explanations. Problem solving, which is essentially a creative activity, cannot be built exclusively on routines, recipes, and formulas.
- The mathematics teacher should assist the student to read and understand problems presented in written form, to hear and understand problems presented orally, and to communicate about problems in a variety of modes and media.
- The mathematics curriculum should provide opportunities for the student to confront problem situations in a greater variety of forms than the traditional verbal formats alone; for example, presentation through activities, graphic models, observation of phenomena, schematic diagrams, simulation of realistic situations, and interaction with computer programs.

1.4 *Appropriate curricular materials to teach problem solving should be developed for all grade levels.*

- Most current materials strongly emphasize an algorithmic approach to the learning of mathematics, and as such they are inadequate to support or implement fully a problem-solving approach. Present textbook problems tend to be easily categorized and stylized and often bear little resemblance to highly diversified, real-life problems. They do not permit the full range of strategies and abilities actually demanded in realistic problem contexts.
- The potential of computing technology for increasing problem-solving ability should be explored and exploited by the development of creative and imaginative software.

1.5 *Mathematics programs of the 1980s should involve students in problem solving by presenting applications at all grade levels.*

- Applications should be presented that use the student's growing and changing repertoire of basic skills to solve a multitude of routine and commonplace problems essential to the day-to-day living of every citizen.
- Applications of mathematics to other disciplines such as the social sciences, business, engineering, and the natural sciences should be presented.
- The enormous versatility of mathematics should be illustrated by presenting as diversified a collection as possible at the given grade level.
- At the college level, courses in mathematics and the mathematical sciences should give prospective teachers experiences that develop their capacities in modeling and problem solving.

1.6 *Researchers and funding agencies should give priority in the 1980s to investigations into the nature of problem solving and to effective ways to develop problem solvers.*

- Support should be provided for
  - —the analysis of effective strategies;
  - —the identification of effective techniques for teaching;
  - —new programs aimed at preparing teachers for teaching problem-solving skills;
  - —investigations of attitudes related to problem-solving skills;
  - —the development of good prototype material for teaching the skills of problem solving, using all media.

Recommendation 2: The concept of basic skills in mathematics must encompass more than computational facility

There must be an acceptance of the full spectrum of basic skills and recognition that there is a wide variety of such skills beyond the mere computational if we are to design a basic skills component of the curriculum that enhances rather than undermines education.

We recognize as valid and genuine the concern expressed by many segments of society that basic skills be a part of the education of every child. However, the full scope of what is basic must include those things that are essential to meaningful and productive citizenship, both immediate and future.

The agreement among parents, educators, and mathematicians on the need for teaching basic skills with greater effectiveness unfortunately does not yet extend to a common understanding and acceptance of exactly what these basic skills should comprise.

Some groups narrowly limit them to routine computation at the expense of understanding, applications, and problem solving. This would leave little hope of developing the functionally *competent* student that all desire.

It must also be recognized that individual capacities, interests, and future directions might call for different emphases and different selections in matching basic skills to individual needs.

The time and energy that teachers and programs should be devoting to building beyond minimal foundations are sometimes skirted, being considered risky deviations from the minimal targets on which educators believe they will be judged. There is great pressure today to use all such time, energy, and resources on overkill in the minimal target areas even though little added productivity may be achieved.

Rather than fostering a return to some acceptable common threshold of performance, the back-to-basics movement tends to place a low ceiling on mathematical competence—and this at the onset of an era in which daily life will be more deeply permeated by multiple and diverse uses of mathematics than ever before. Under these circumstances, even if improvement in rote computation takes place, a citizen who cannot analyze real-life situations to the point of recognizing what computations must be made to solve real-life problems has not entered the mainstream of functional citizenship.

It is dangerous to assume that skills from one era will suffice for another. Skills are tools. Their importance rests in the needs of the times. Skills once considered essential become obsolete, and this is likely to increase in pace and scope as advances in technology revolutionize our individual, social, and economic lives. Necessary new skills arise from the dimensions of the mathematics pertinent to an age of population explosion, space exploration, economic and fiscal complexity, and microelectronic wonders. Time and space for including these new skills in the curriculum must be purchased by eliminating the obsolete.

Insisting that students become highly facile in paper-and-pencil computations such as 3841 × 937 or 72,509 ÷ 29.3 is time-consuming and costly. For most students, much of a full year of instruction in mathematics is spent on the division of whole numbers—a massive investment with increasingly limited productive return. A small fraction of that time is spent on the skills of problem analysis and interpretation, which enable students to identify and set up the computations needed. For most complex problems, using the calculator for rapid and accurate computation makes a far greater contribution to functional competence in daily life.

Common sense should dictate a reasonable balance among mental facility with simple basic computations, paper-and-pencil algorithms for simple problems done easily and rapidly, and the use of a calculator for more complex problems or those where problem analysis is the goal and cumbersome calculating is a limiting distraction.

Reasonable standards of time-effectiveness and cost-effectiveness should be applied to the use of instructional time, where the criterion is the productive applicability of the learned technique to real-life problems.

Professional knowledge of future trends, industrial, financial, engineering, and scientific need, and the demands of daily life are all better arbiters of what is currently essential and what has become obsolete than our nostalgia as parents or teachers.

Recommended Actions

2.1 *The full scope of what is basic should contain at least the ten basic skill areas identified by the National Council of Supervisors of Mathematic's "Position Paper on Basic Skills." These areas are problem solving; applying mathematics in everyday situations; alertness to the reasonableness of results; estimation and approximation; appropriate computational skills; geometry; measurement; reading; interpreting and constructing tables, charts, and graphs; using mathematics to predict; and computer literacy.*

2.2 *The identification of basic skills in mathematics is a dynamic process and should be continually updated to reflect new and changing needs.*

2.3 *Changes in the priorities and emphases in the instructional program should be made in order to reflect the expanded concept of basic skills.*

- There should be increased emphasis on such activities as
  - —locating and processing quantitative information;
  - —collecting data;
  - —organizing and presenting data;
  - —interpreting data;
  - —drawing inferences and predicting from data;
  - —estimating measures;
  - —measuring using appropriate tools;
  - —mentally estimating results of calculations;
  - —calculating with numbers rounded to one or two digits;
  - —using technological aids to calculate;
  - —using ratio and proportion to deal with rate problems in general and with percent problems in particular;
  - —using imagery, maps, sketches, and diagrams as aids to visualizing and conceptualizing a problem;
  - —using concrete representations and puzzles that aid in improving the perception of spatial relationships.
- There should be decreased emphasis on such activities as
  - —isolated drill with numbers apart from problem contexts;
  - —performing paper-and-pencil calculations with numbers of more than two digits;
  - —mastering highly specialized vocabulary not useful later either in mathematics or in daily living;
  - —converting measures given in one system to corresponding measures in another system;
  - —working with tables whose usefulness as aids to calculation has been supplanted by calculators and other technological aids (e.g., numerical computations with logarithms and cologs).

2.4 *Teachers should incorporate estimation activities into all areas of the program on a regular and sustaining basis, in particular encouraging the use of estimating skills to pose and select alternatives and to assess what a reasonable answer may be.*

2.5 *Teachers should provide ample opportunities for students to learn communication skills in mathematics. They should systematically guide students to read mathematics and to talk about it with clarity.*

2.6 *The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language, including*

- —the identification of likenesses and differences leading to classification;
- —understanding, making, and applying definitions;
- —the development of a feeling for informal proof including counter-examples and generalizations;
- —precise use of such language as *at least, at most, either-or, both-and,* and *if-then.*

Recommendation 3: Mathematics programs must take full advantage of the power of calculators and computers at all grade levels

Beyond an acquaintance with the role of computers and calculators in society, most students must obtain a working knowledge of how to use them, including the ways in which one communicates with each and commands their services in problem solving.

The availability of computing aids, including computers and calculators, requires a reexamination of the computational skills needed by every citizen. Some of these computational skills will no longer retain their same importance, whereas others will become more important.

It is recognized that a significant portion of instruction in the early grades must be devoted to the direct acquisition of number concepts and skills without the use of calculators. However, when the burden of lengthy computations outweighs the educational contribution of the process, the calculator should become readily available.

With the increasing availability of microcomputers at decreasing costs, it is imperative that schools play an active part in preparing students of the 1980s to live in a world in which more and more functions are being performed by computers.

Recommended Actions

3.1 *All students should have access to calculators and increasingly to computers throughout their school mathematics program.*

- Schools should provide calculators and computers for use in elementary and secondary school classrooms.
- Schools should provide budgets sufficient for calculator and computer maintenance and replacement costs.

3.2 *The use of electronic tools such as calculators and computers should be integrated into the core mathematics curriculum.*

- Calculators should be available for appropriate use in all mathematics classrooms, and instructional objectives should include the ability to determine sensible and appropriate uses.
- Calculators and computers should be used in imaginative ways for exploring, discovering, and developing mathematical concepts and not merely for checking computational values or for drill and practice.
- Teachers should ensure in their classroom management that the use of computers by individual students in isolated activity does not replace the critical classroom interaction of students with peers and teacher. The healthy give-and-take of group work and discussion, which promotes values of communication, cooperation, empathy, mutual respect, and much of cognitive development, remains essential.

3.3 *Curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available.*

- Schools should insist that materials truly take full advantage of the immense and vastly diverse potential of the new media. In particular, developers of software should be cautioned that just to use conventional material and techniques newly translated to the medium of the computer will not suffice.
- Educators should take care to choose software that fits the goals or objectives of the program and will not twist the goals and developmental sequence to fit the technology and available software.

3.4 *A computer literacy course, familiarizing the student with the role and impact of the computer, should be a part of the general education of every student.*

- In cooperation with schools and professional teacher organizations, funding agencies should support the development of courses in computer literacy for both junior and senior high school levels.

3.5 *All mathematics teachers should acquire computer literacy either through preservice programs or through inservice programs funded by school districts in order to deal with the impact of computers on their own lives and to keep pace with the inevitable sophistication their students will achieve.*

- Colleges should provide courses for both preservice and inservice education in computer literacy, programming, and instructional use of calculators and computers.

- Professional organizations should provide information through their various media, conferences, workshops, and seminars to aid in the inservice education of teachers in uses of the calculator and computer.

3.6 *Secondary school computer courses should be designed to provide the necessary background for advanced work in computer science.*

- Curriculum design should provide the required foundation for those students who will be involved in careers that increasingly demand advanced computing skills and applications of computing and for those students who will go on to deeper study in frontier fields of computer development.

3.7 *School administrators and teachers should initiate interaction with the home to achieve maximum benefit to the student from the coordinated home and school use of computers and calculators.*

- Criteria should be developed to assist parents and school personnel in their selection of home/school computing hardware.
- Professional organizations of teachers, mathematicians, and computer scientists should develop guidelines to aid schools, teachers, and parents in the selection of educational software.
- The uses of technological devices such as calculators, computers, video disks, and electronic games in the home and other out-of-school places should be anticipated. Programs should be planned that will encourage the positive and educationally beneficial use of these devices.
- As home computers come into wider use, homework should be assigned that can take advantage of their potential in problem solving.

3.8 *Educational users of electronic technology should demand a dual responsibility from manufacturers: the development of good software to promote the problem-solving abilities of the student and, eventually, the standardization and compatibility of hardware.*

3.9 *Provisions should be made by educational institutions and agencies to help in the necessary task of educating society's adults in computer literacy and programming.*

3.10 *Teachers of other school subjects in which mathematics is applied should make appropriate use of calculators and computers in their instructional programs.*

3.11 *Teacher education programs for all levels of mathematics should include computer literacy, experience with computer programming, and the study of ways to make the most effective use of computers and calculators in instruction.*

3.12 *Certification standards should include preparation in computer literacy and the instructional uses of calculators and computers.*

Recommendation 4: Stringent standards of both effectiveness and efficiency must be applied to the teaching of mathematics

What is learned relative to a topic, how long it is retained, how readily it is applied—all these depend on the learning process the students pass through and how effectively they are engaged in that process. It is fruitless to consider topics taught apart from the way learners meet these topics.

Instructional time is a precious commodity. It must be spent wisely. Learning is a product of both the time engaged in a learning task and the quality of that engagement. Teachers must employ the most effective and efficient techniques at their command. They must apportion instructional time according to the importance of the topic, recognizing that the value of a skill or knowledge is subject to change over time.

Modern technology and educational theory and research have made accessible to today's teacher approaches, materials, and strategies that were not previously available. Teachers at all levels must learn to use this enriched variety of instructional techniques, materials, and resources to teach mathematics more effectively.

Recommended Actions

4.1 *The major emphasis on problem solving in the curriculum must be accommodated by a reprogramming of the use of time in the classroom.*

- Priority in classroom time should be devoted to involving students in meaningful problem-solving activities. Explanation, practice, and directive teaching are important but should not diminish the time necessary to achieve this priority. Requiring complete mastery of skills before allowing participation in challenging problem solving is counterproductive.
- The time spent on mathematics in elementary school programs should be increased. Higher-order skills in problem solving require more time to learn than the lower-order, narrowly mechanistic skills.
- The extent of teacher and student time devoted to certain traditional skill areas should be reduced to make room for newly emerging objectives.
- There are certain algorithmic skills (e.g. long division with multiple-digit divisors) that require a great expenditure of classroom time. A strict standard of time-effectiveness and cost-effectiveness should be applied to determine whether actual use of that technique in life outside school justifies this much expenditure of effort and time. The use of calculators has radically reduced the demand for some paper-and-pencil techniques.
- Teachers should learn effective techniques of classroom management to assess and achieve the optimal time on a task.

4.2 *School administrators and parents must support the teacher's efforts to engage students more effectively in learning tasks.*

- Local administrators should ensure uninterrupted time for the teacher to carry out the instructional program.
- Parents and administrators must support the authority of the teacher to require that students be productively engaged in learning during their class time. They should exercise reasonable sanctions against students who do not respect that authority.
- Parents and administrators should expect that teachers will assign a reasonable amount of meaningful homework to extend productively the time students are engaged in the study of mathematics. Both should use their influence to increase the likelihood that students will complete the homework as assigned.
- School budgets should provide for a range of instructional resources adequate to support a wide variety of teaching strategies.

4.3 *Teachers should use diverse instructional strategies, materials, and resources, such as*

— individual or small-group work as well as large-group work;

— well-planned use of media, such as overhead projectors, videotapes, video disks, audio/video cassettes, computers, films, slides, television;

— the provision of situations that provide discovery and inquiry as well as basic drill;

— the use of manipulatives, where suited, to illustrate or develop a concept or skill;

— the inclusion of cyclic review of past topics (contents, skills, and ideas previously taught);

— the use of materials and references outside the classroom, such as visiting museums, using the library, visiting businesses or industries, visiting computer centers, making home television assignments.

4.4 *School districts, local and state or provincial officials, manufacturers, and publishers should take a bolder and more imaginative approach to selecting and producing educational hardware and software in order to provide for a curriculum that emphasizes problem solving.*

- Unnecessarily restrictive conditions set by school districts, state and provincial agencies, outdated ground rules of publishing houses, conservative status quo editorial policies, and the like should give way to greater openness and willingness to address future needs.

Recommendation 5: The success of mathematics programs and student learning must be evaluated by a wider range of measures than conventional testing

The first purpose of meaningful evaluation in school mathematics should be the improvement of learning programs, teaching, and materials. Educators must evaluate to have information for sound decisions, to be accountable to their public, and to know how well they are doing. Evaluation is a part of mathematics teaching, and hence mathematics educators should be centrally involved in the evaluation process.

Evaluation is not limited to testing. It includes gathering data and interpreting the data. Testing is one source of data. There are many others. Today, many people use test scores as the sole index of the quality of mathematics programs or of the success of student achievement. Test scores alone should not be considered synonymous with achievement or program quality. A serious danger to the education of our youth is the increasing tendency on the part of the public to assume that the sole objective of schooling is a high test score. This is often assumed without the critical knowledge of what is being tested or whether test items fit desired goals.

The evaluation of problem-solving performance will demand new approaches to measuring. Certainly present tests are not adequate. In particular, the measuring of the use of problem-solving processes will demand innovative techniques. Evaluation of programs with problem-solving goals must be sensitive to the nature of those goals. It is imperative that the goals of the mathematics program dictate the nature of the evaluations needed to assess program effectiveness, student learning, teacher performance, or the quality of materials. Too often the reverse is true: the tests dictate the programs, or assumptions of the evaluation plan are inconsistent with the program's goals.

Recommended Actions

5.1 *The evaluation of mathematics learning should include the full range of the program's goals, including skills, problem solving, and problem-solving processes.*

- The evaluation of the use of problem-solving processes must be given special attention by schools, teachers, researchers, test developers, and teacher educators.
- Assessment programs, such as the National Assessment and some state assessments, should continue to sample a wide variety of mathematics learning outcomes and should consider future as well as present needs and programs.
- The development of problem-solving skills should be assessed for each student over the entire K-12 school mathematics program.
- Minimal competencies should not be construed as an adequate measure of an individual's mathematic achievement. What is minimal for all is optimal for none.

5.2 *Parents should be regularly and adequately informed and involved in the evaluation process.*

- With mutual respect (the educator for the sincere concern and valid input of the parent and the parent for the professional expertise of the educator), school administrators, teachers, and parents should cooperate in determining educational goals and the appropriate plan for evaluation.
- Nontest evaluation methods and strategies should be discussed with parents, students, and the general public.

5.3 *Teachers should become knowledgeable about, and proficient in, the use of a wide variety of evaluative techniques.*

- Preservice and inservice teacher education should provide mathematics teachers with knowledge about, and skill in, evaluation.
- Evaluation strategies that include both test and nontest techniques should be developed and disseminated to mathematics teachers both in their initial preparatory program and in continual inservice updating.
- Teachers should improve their diagnostic skills and their ability to structure appropriate remediation.
- Student strengths and weaknesses should be assessed on a regular basis, using a variety of measures and techniques.
- The informed judgment of teachers should be considered a vital part of the evaluation of any student.
- Teachers should be prepared in the development and use of teacher-made tests.

5.4 *The evaluation of mathematics programs should be based on the program's goals, using evaluation strategies consistent with these goals.*

- Standardized tests should be used in program evaluations only when it can be clearly demonstrated that the test matches the goals of the program.
- The results of a test designed for purposes other than program evaluation (such as the SAT) should

not be interpreted as an evaluation of mathematics programs.

- Available evaluation and testing techniques should not determine the goals and objectives of the mathematics program or the emphases of classroom instructional effort.
- Test designers should give attention to the need for more options in format than the conventional multiple-choice format. An emphasis on problem solving demands more flexibility and creativity in assessment than is possible within the restrictions of most current test formats.
- Where minimal competency tests are mandated, they should be implemented with extreme caution to assure that adverse effects on the program do not result.
- Task forces involving parents, teachers, and students should be created to monitor the effects of minimal competency programs.
- Mathematics educators should be centrally involved in the development of competency or assessment programs at local, state, and provincial levels and in the monitoring of the effects of competency testing on mathematics learning.
- Longitudinal evaluation of individual problem-solving ability should be developed. The acquisition of problem-solving skills is a long-term process and should not be evaluated solely with short-term measures.
- Test scores should not be used as the sole index of success in mathematics programs.
- Accreditation of school mathematics programs should use criteria specific to the quality of the mathematics programs rather than to conditions peripheral to content and instructional goals.

5.5 *The evaluation of materials for mathematics teaching should be an essential aspect of program planning.*

- Textbook materials should be judged and selected in terms of the program's goals rather than vice versa.
- Instructional materials with sexist and ethnic biases should not be selected.
- The selection of tests should involve a careful review by teachers as well as administrators.
- Strategies for evaluating nonprint materials must be developed and used.

5.6 *Mathematics teachers must undergo continuing evaluation as a necessary component in improving mathematics programs.*

- Teachers should maintain an awareness of the need and the strategies for self-evaluation.
- A variety of supportive evaluation strategies, such as peer observation, supervisor observation, and videotaping, should be made available to the teacher.
- Any evaluation of mathematics teaching should be sensitive to the instructional goals and should be unique to the content, the teacher, and the class.
- If an evaluation of a teacher's effectiveness includes student performance, the measures of student performance should be consistent with instructional goals. Such evaluation should also consider that there are external factors affecting student performance that are not amendable to teacher influence.
- Judgments of teacher competency are necessary but should be made with caution, with the realization that the validity of most existing measures rests on a shaky foundation.
- Teachers and teacher educators should be centrally involved in the development of instruments for the evaluation of teaching effectiveness.

5.7 *Funding agencies should support research and evaluation of the effects of a problem-solving emphasis in the mathematics curriculum.*

- The nature of problem-solving ability suggests that longitudinal studies will be most meaningful. The more typical short-term project may force a hasty and superficial treatment of programs whose objectives must be complex, interrelated, and of a long-lasting character.

Recommendation 6: More mathematics study must be required for all students and a flexible curriculum with a greater range of options should be designed to accommodate the diverse needs of the student population

Mathematics is pervasive in today's world. Mathematical competence is vital to every individual's meaningful and productive life. It is, moreover, a valuable societal resource, and the potential of our educated citizenry to make significant use of mathematics is not being fully met. The amount of time the majority of students spend on the study of mathematics in school in no way matches the importance of mathematical understanding to their lives, now and in the future.

Polls show that the public recognizes that mathematics is an essential subject for all students. Yet it is typical to find that only one year of mathematics is required for students in grades 9-12. Surveys of instructional time indicate that only a small percentage of time in elementary school is devoted to mathematics. And only a very tiny portion of that time is spent on teaching children to apply mathematics and solve problems.

When a student discontinues the study of mathematics early in high school, he or she is foreclosing on many options. Many doors, both in college programs and in vocational training, are at once closed to that person. These facts should be communicated more effectively to both the students and their parents.

The recommendation for increasing mathematics study imposes a special responsibility on professional leadership in mathematics education and the schools. To say that most students should study more mathematics is not to say that it should be the same mathematics for all. It does not mean simply keeping all students longer in the same traditional track. In fact, such a recommendation poses a tremendous challenge to curriculum developers and school districts to devise a more flexible range of options, a diversified program to meet a variety of interests, abilities, and goals.

Except for a distinction among courses typically taken in the ninth grade, differentiation in students' mathematics programs has been based mainly on whether or not a student takes elective years of mathematics. If everyone now needs more mathematics study, then differentiation must occur within the program.

The growing diversification of the applications of mathematics in an ever-increasing variety of college programs of study demands more than a single college-preparatory program. At the college level, the trend is to broaden the conception of mathematical study to include what is called "the mathematical sciences." Roughly, this means not only mathematics but the delivery of mathematical ideas and tools to the solution of real-life problems. Present high school programs do not fully anticipate the many options provided by mathematics, the mathematical sciences, and computer science.

Technical and vocational training at different levels also assumes more and diverse mathematical backgrounds. For those whose formal education will end with high school, the needs of citizen and consumer for increasing mathematical sophistication dictate a collection of courses based on consumer and career needs, computer literacy, and quantitative literacy.

It is important that recommended programs permit lateral movement and not strictly "track" students, trapping them in a linear pattern that does not permit change to another path. Flexibility is vital, and the key is to keep options open as long as possible.

Since a higher level of mathematical skill and understanding will increasingly become a significant advantage in nearly all lives, then justice demands that all groups have equal access to these advantages. At present, females and some minority groups are underrepresented in mathematics courses and courses for which mathematics is prerequisite.

It is naive to suppose that just providing mathematics courses as electives will serve for equality of opportunity. All reasonable means should be employed to assure that everyone will have the foundation of mathematical learning essential to fulfilling his or her potential as a productive citizen. The currently underrepresented groups should be especially encouraged and helped.

Recognizing diversified individual interests and needs entail devising programs that are tailored for particular categories of students. Differentiated curricula must incorporate the special needs in mathematics of students with handicaps, including physical or learning difficulties. These programs will need to move away from the idea that everyone must learn the same mathematics and develop the same skills. Mathematics and mathematical ability cover a much broader range than most people realize.

In many current programs, the student who does poorly in the algorithmic skills finds progress in all aspects of mathematical development halted, since remediation is designed to concentrate solely on this deficiency. Remedial programs should identify other areas of mathematical ability—for example, spatially related skills—and concentrate attention also on the students' strengths, not solely on their deficits.

The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world.

Mathematics educators and curriculum developers should redesign the sequence of the curriculum to realize the critical process goals of problem solving, as well as content and skill goals. A clear and logical developmental sequence for process objectives from kindergarten through twelfth grade should be described and serve as an organizing framework.

In very general terms, such a sequence should proceed through stages of development, though in practice, progression should be smooth and unbroken. In the elementary school, children, in their experiences in solving particular problems, gradually develop their higher-order mental abilities and learn particular skills and particular strategies. At this stage, strategies are primarily specific to individual problems. A foundation of skills must be laid, but skills should not be learned entirely in isolation from application even in the primary grades. There should be an interplay of skill building and application throughout.

Moving through the upper grades and into the junior high school level, the progress should be toward more generalization, more abstraction of techniques, more emphasis on similarities and patterns found in differing contexts. Strategies become not just ways to solve one type of problem but generalized and synthesized. Techniques learned in one context may be recognized as applicable to other problems. Specific attention is needed to help students make the important transition to the abstract reasoning processes.

During the seventh and eighth grades, intensive focus on problem solving should become a vehicle to exercise, confirm, and develop further all basic skills. At the same time, familiarity, competence, and confidence should be built in applying these mathematical skills to solving problems of varying difficulty and from diverse settings. At this stage, a significant skill is the ability to select strategies from a growing repertoire.

A broad range of problem-solving approaches should be explored so that the teacher can identify students' special strengths and advise them on high school options.

The ability to create strategies to attack a new problem is, at the junior high school level, simple and embryonic. It should increase in sophistication throughout high school. More flexibility and power should be achieved by more generalization and abstraction. The essential nature of techniques and strategies and their range of applicability should be emphasized as students begin to see more applications to broad categories of problem areas, as in, for example, application to other disciplines. There should be a greater effort made to coordinate the mathematics learned in mathematics classes with other subjects that use mathematics. Students who perform well in mathematics classes often fail to see any relationship in what they are doing there to the mathematical techniques employed in classes in science and other subjects.

This does *not* suggest that the content of high school mathematics is either dictated by or limited to the range of mathematical techniques used in other subjects. The goal is to develop a more flexible, deeper, and broader problem-solving power, and this goes significantly beyond the formulas and recipes that have been traditionally applied to familiar problems.

At the same time, the important interplay and integration of mathematics and its applications in learning should not cease because isolated course structure separates mathematics from disciplines that apply it. The principles of learning must take priority over administrative convenience. It is likely that this coordination will have to be accomplished by voluntary cooperative efforts on the part of mathematics teachers and the teachers of other subjects that apply mathematics, with a healthy mutual respect for the legitimate goals of both groups.

At the high school level many students can apply their problem-solving abilities not only to problems of daily life, to problems from other disciplines, but to serious mathematical problems themselves.

Recommended Actions

6.1 *School districts should increase the amount of time students spend in the study of mathematics.*

- At least three years of mathematics should be required in grades 9 through 12.
- The amount of time allocated to learning mathematics in elementary school should be increased. It should range from a minimum of about five hours a week in the primary grades to a minimum of about seven hours a week in the upper grades. Part of this time can be gained by a program that stresses the application of mathematical skills in other subjects, especially the sciences and the social studies.
- A clear delineation of what constitutes college-level and precollege mathematics should be made.
- Colleges should not award college credit for courses in which the level of content is that of high school mathematics. This practice encourages students not to elect mathematics in high school beyond the minimum required.

6.2 *In secondary school, the curriculum should become more flexible, permitting a greater number of options for a diversified student population.*

- Increasing high school requirements in mathematics should not result in keeping all students longer in the same traditional tracks. These recommendations cannot be met with just the two-track alternatives of either general mathematics courses or precalculus courses typical of many existing programs.
- The high school curriculum should provide differing student populations with those appropriately organized areas of mathematical competence required by their needs, talents, and future objectives, but all presented with continual attention to functional problem-solving ability.
- Algebra should be included in the program of all capable students to keep their options open.
- For many students, algebra should be delayed until a level of maturity and basic mathematical understanding permit their taking full advantage of a significant algebra course. For many, this may not be ninth grade but perhaps eleventh or twelfth grade. Significant mathematics courses should be available to these students in ninth and tenth grades, not just the traditional general mathematics review or prealgebra courses.
- If such recommendations are followed, a course providing progress beyond junior high school but paving the way for a useful experience in consumer mathematics and later algebra needs to be developed.
- Consumer mathematics should develop a broader quantitative literacy and should consist primarily of work in informal statistics, such as organizing and interpreting quantitative information.
- All high school students should have work in computer literacy, and the hands-on use of computers, and the applications of computers where possible and appropriate throughout their mathematics programs.
- All students who plan to continue their study of mathematics beyond high school or to use it extensively in technical work or training should be enrolled in mathematics courses throughout their last high school year.

6.3 *Mathematics educators and college mathematicians should reevaluate the role of calculus in the differentiated mathematics programs.*

- Emerging programs that prepare users of mathematics in nontraditional areas of application may no longer demand the centrality of calculus that has traditionally been demanded for all students. (The Mathematical Association of America's PRIME 80 conference raised questions about the role of calculus as the eventual touchstone that dictated all college preparatory mathematics.)
- In light of this reevaluation, colleges and high schools should reexamine the concept of advanced placement in mathematics. For some students, though fully capable, an advanced placement program restricted to calculus placement may not be the optimal alternative. If advanced placement in mathematics is encouraged, it should be a broader concept that includes options in other branches of the mathematical sciences.

6.4 *The curriculum that stresses problem solving must pay special heed to the developmental sequence best suited to achieving process goals, not just content goals.*

- From the earliest years, the basic mathematical tools should be acquired within the framework of usage and application, however simple the examples may be at the beginning level.

- Since there are usually multiple approaches to all but the most trivial problems, not all students should be expected to proceed in the same way. Value should be placed on a thoughtful and productive approach, not solely on a single correct answer.
- Since elementary school children differ widely in maturation and intellectual development, the teacher should be prepared to value and reward different contributions made by different students to the solution of a common problem.
- Team efforts in problem solving should be commonplace in the elementary school classroom.
- At middle school and junior high school levels, the focus of the curriculum should be on more formal and more general problem-solving approaches and strategies themselves.
- At middle school and junior high school, instruction should stress the ability to apply techniques used in one situation to new and unfamiliar situations.
- At middle school and junior high school, instruction should stress the ability to select from a range of strategies and to create new strategies by combining known techniques.
- At middle school and junior high school, instruction should aid in the student's transition to more abstract reasoning.
- Difficulty with paper-and-pencil computational algorithms should not interfere with the learning of problem-solving strategies.
- At the junior high school level, calculators should be available so that no student will be excluded from the opportunity to develop these strategies.
- All courses in high school mathematics should include activities in applications.
- Teachers of mathematics and teachers of other disciplines should cooperate in assuring that students perceive the relationship of the mathematics they learn to the mathematics applied in problems in those other disciplines.
- Qualified mathematics teachers should be used as resource specialists for instructional programs in which mathematics methods are applied in other subjects.
- Teachers of mathematics should be prepared in the application of known problem-solving techniques to a variety of problems.

6.5 *Teachers, school officials, counselors, and parents should encourage a positive attitude toward mathematics and its value to the individual learner.*

- A curriculum that focuses on problem solving should challenge all students. It should present the opportunity for students of all ability levels to make a contribution, and it should promote the attitude that they are capable of solving problems.
- Programs that will encourage a larger percentage of females and minority students to study more mathematics should be designed and supported.
- Parents and counselors should help students to recognize the importance of mathematics study to their futures and guide them to make appropriate school decisions.

6.6 *Special programs stressing problem-solving skills should be devised for special categories of students.*

- The professionals in mathematics and in special education should work together to identify the process abilities, possible and optimal, for students with handicaps and learning difficulties.
- Perception of what mathematical ability encompasses should be broadened beyond the linear, algorithmic stereotype. The significance of spatial perception and spatial relationships in problem solving needs to be stressed, for many students who do not fare well in algorithmic thinking may have special abilities that are spatial and geometric.
- Increased attention should be paid to developing the potential of the gifted student of mathematics.
- Colleges and schools should cooperate in devising imaginative programs for the mathematically gifted.
- In general, programs for the gifted should be based on a sequential program of enrichment through more ingenious problem-solving opportunities rather than through acceleration alone.
- Materials and resources of a sophistication and depth suitable to the unusual potential of the gifted student in problem solving should be developed.

Recommendation 7: Mathematics teachers must demand of themselves and their colleagues a high level of professionalism

Educators can gain the support of society and the rewards of a truly effective performance by developing, defining, and enforcing professional standards in terms of highly competent professional performance rather than by any other norm. This must be done to provide the nation, its young people, and its future with the mathematics programs worthy of them and of that future.

Within mathematics teaching

- there are already many well-prepared and effective teachers who provide outstanding professional leadership.
- there are also many teachers who are motivated and desire to improve but who lack adequate preparation and must be given the necessary support to become fully qualified and to improve;
- there are, however, some teachers whose attitudes and functioning are at less than a professional level. In the best interests of students and society, the number of such teachers must be reduced immediately.

Even the best prepared, competent, and dedicated teachers must continue their development to keep abreast of changing needs, tools, and conditions. School administrators have the responsibility to make this possible by providing continuing inservice education and by encouraging teachers to take full advantage of opportunities for maintaining their competence that are offered by professional organizations and universities.

At the beginning of this decade, the schools are faced with a widespread shortage of qualified mathematics teachers. The demand for mathematical competence in many sectors of society is great and growing, and schools find it impossible to compete for individuals who have this desired background. Thus, in many mathematics classrooms, the teacher does not have the subject-matter qualifications for teaching mathematics. School administrations have two obligations: (1) to be forthright and open with the parents about the situation, and (2) to provide special aid and support needed by these teachers until they can make up deficiencies. The professional organizations also have a special responsibility to cooperate in assisting such teachers who are dedicated to improvement. However, the public and its representatives must give high priority to finding ways to solve the worsening problem.

Regardless of preparation, the standard of professionalism should be consistently high, and it is the obligation of any group that wishes to be called a profession to insist that all members maintain this standard.

Teachers must be sensitive to the needs of their students and dedicate themselves to the improvement of student learning as their primary professional objective. The right of students and parents to expect this dedication has correlative responsibilities: a mutual respect and support by the parent of the educational program and a teacher's professional competence, and the acceptance by the student of ultimate personal and active commitment to his or her own learning.

Any teacher who lacks dedication to these professional ideals and to continued self-improvement should not be retained in teaching. Teachers must accept performance and not protectionism as a synonym for professionalism.

During the decade of the 1980s, the continuing appearance of new concepts and theories in mathematics, in the applications of mathematics, and in the teaching-learning process will affect both curriculum and instruction in school mathematics. In order to remain professional, teachers must continue to study in all three areas. This will require of teachers a new level of motivation and dedication.

Recommended Actions

7.1 *Every mathematics teacher should accept responsibility for maintaining teaching competence.*

- Full advantage should be taken of all existing opportunities for continuing education.
- Teachers should insist that school districts and colleges make provision for inservice education and staff development opportunities.
- Teachers should belong to professional organizations that are dedicated to the improvement of teaching and learning.
- Teachers should participate actively in the efforts of professional organizations to improve teaching and learning.

- Teachers should share ideas and participate with their peers in cooperative efforts at self-improvement, including observation and constructive criticism of one another.
- Teachers as a profession should insist that all members maintain a consistently high standard of professional behavior. The profession is not obligated to protect those individuals who refuse to live up to reasonable professional standards.
- Bargaining units of organizations representing teachers should include in their requests release time for professional development and attendance at professional conferences that provide inservice education.

7.2 *School boards and school administrators should take all possible means to assure that mathematics programs are staffed by qualified, competent teachers who remain current in their field.*

- Necessary incentives must be found to attract competent and dedicated teachers to the profession.
- The status, compensation, and teaching conditions necessary for the retention of qualified teachers must be dramatically improved.
- School districts must budget adequately and provide incentives for teachers to participate in inservice education pertinent to their immediate needs as they prepare to meet the challenges of the future.
- School district staffs, including teachers, must plan inservice education that is articulated with local colleges and universities as well as with professional organizations such as the NCTM and its state and local affiliates.
- School administrators should encourage teachers to take an active professional role and should permit them to participate, without penalty, in conferences and vital professional work.
- School systems should maintain well-qualified mathematics specialists or supervisors at all levels to help teachers achieve the professional level specified in these recommendations and to coordinate mathematics education efforts within the system.
- Well-qualified mathematics specialists should be on the staffs of all governmental agencies that deal with mathematics education—specifically, in all state and provincial departments of education, in the National Education Department, the National Institute of Education, and the National Science Foundation.

7.3 *Teacher education institutions and agencies should develop new programs of preparation to incorporate the problem-solving emphases recommended.*

- Programs in teacher education must be designed to prepare teachers for new levels of performance and professionalism.
- The effective teaching of problem solving requires thorough preparation both in mathematical content and in teaching methods that develop problem-solving ability.
- Colleges and universities must redesign courses and programs to incorporate revisions recommended for the preparation of mathematics teachers.

7.4 *Certification standards for mathematics teaching should be revised and upgraded to incorporate the needs reflected in these recommendations.*

- Professional organizations should participate in defining standards for certification of teachers.
- State and provincial departments of education should involve professional organizations in setting professional standards and defining qualifications.
- Professional organizations—such as the Mathematical Association of America, Conference Board of the Mathematical Sciences, National Council of Supervisors of Mathematics—in cooperation with the NCTM should continually review and update their guidelines for the preparation of teachers of mathematics.

Recommendation 8: Public support for mathematics instruction must be raised to a level commensurate with the importance of mathematical understanding to individuals and society

Solutions to the problems identified in previous sections cannot be achieved solely within the education community but require active participation and support by parental and societal groups.

The mathematics teaching profession recognizes and respects the right of parents and society to hold it accountable for the mathematical competence of children. However, in calling for particular programs of action, parents and society often mistakenly promote activities that are counterproductive to the realization of the goals they support. Communication and cooperation must bridge this gap.

The rededication of teachers called for in Recommendation 7 will be meaningless or impossible if society is not committed to supporting the professionalism it rightfully demands. Today, teachers' financial incentives are neither commensurate with the responsibilities they carry nor adequate to attract and retain them in a demanding, crucial, and sometimes burdensome profession. Few school systems have truly adequate supervisory and material support for the teacher in the maintenance and improvement of the instructional environment. More and more, teachers feel the lack of parental understanding of the complexity of their task and the lack of parental cooperation and support in their efforts to instruct children. Furthermore, governmental support for improving the quality of mathematics teaching has dwindled.

Essential to the success of the program outlined in these recommendations are (1) the willingness of government and private funding agencies to listen and respond more sensitively to the professionals in elementary and secondary mathematics education, and (2) the reestablishment by such agencies of institutes and other programs for the continuing education and upgrading of teachers.

At present, there are too many unnecessary obstacles to the effective functioning of teacher and student in a true teaching/learning interaction. These include more and more time required for unproductive recordkeeping; many unmotivated, undisciplined students; a lack of parental support; ambivalence and vacillation in government regulations; shifting societal priorities; the lack of home and school agreement on out-of-school study assignments.

There are also obstacles faced by school administrators and by the policymaking bodies, the school boards, whose ultimate responsibility is the effectiveness of the school district. It is recognized that in many instances confusions of legalisms and court decisions impose constraints on the ability of these people to accomplish what they agree should be done. Society today is troubled with many complexities. All institutions are threatened. These obstacles should not be an excuse for inaction; rather, in the light of pressing student needs it is imperative that they be surmounted.

All these things make the ideal professional role of the teacher almost impossible; they make it difficult to attract and retain teachers of a level of competence necessary to realize the outlined program. The immediate loser is the individual student, who is excluded from full participation in, and contribution to, society. The ultimate loser is society itself.

The public and its representatives need to confront a serious problem that is increasing in magnitude. There are not enough qualified mathematics teachers to fill mathematics classroom positions. Present enrollments in teacher preparation programs guarantee that the shortage will increase. Mathematics is of critical importance and should be taught by those well qualified and knowledgeable. Such knowledge is not gained in a crash program of short duration. Parents concerned for their children's future and society concerned for its own future should immediately find extraordinary measures to solve this problem.

The professional community and society share a common goal: to bring all citizens to the full realization of their mathematical capacity. This is a complex and delicate task, and it requires the commitment and cooperation of all segments of society, not just the school, parents, and teachers.

Recommended Actions

8.1 *Society must provide the incentives that will attract and retain competent, fully prepared, qualified mathematics teachers.*

- School districts must provide compensation commensurate with the professionalism and qualifications necessary to achieve our educational goals in mathematics.
- School districts should investigate a variety of incentives and conditions that will stop the drain of qualified mathematics teachers to other, more highly compensated fields of work.
- School districts should assure mathematics teachers a classroom environment and conditions conducive to effective teaching, including a reasonable class size.

- School districts should provide teaching conditions and incentives that will attract dedicated and competent people into the mathematics teaching profession.

8.2 *Parents, teachers, and school administrations must establish new and higher standards of cooperation and teamwork toward the common goal of educating each student to his or her highest potential.*

- Professionals must respond to calls for maintaining educational standards and must work cooperatively with parents to specify these standards.
- Parents must support the maintenance of agreed on standards of achievement and discipline.
- Programs in mathematics that take full advantage of home and school cooperation should be systematically developed.
- Parents should enter the process of determining educational goals as partners, with shared participation and responsibility for the accomplishment of those goals.
- Parents should support the teacher's assignment of homework when it is reasonable and clearly related to the educational objectives.
- Parents should help guide students to an understanding of their critical need to learn mathematics.
- Parents should help students understand that they have the ultimate responsibility for their own learning and must be active and cooperative participants in the process.
- Parents and teachers should cooperate in a mutually supportive attack on the erosion of respect for authority in the classroom.
- Parents and teachers should cooperate in a mutually supportive attack on the erosion of motivation toward academic achievement.

8.3 *Government at all levels should operate to facilitate, not dictate the attainment of goals agreed on cooperatively by the public's representatives and the professionals.*

- Government funding agencies should support an emphasis on research and development in applying mathematics to problem solving.
- Teachers should have advisory roles in all decisions of policy and support.
- When legislation affecting education is required, it should be entered into with caution and only after the involvement of educational professionals in formulating and reviewing the mathematical and pedagogical aspects of such legislation.
- Legislation concerning accountability should take into account the multiple factors that determine school achievement.
- Legislators should avoid tendencies to mandate testing as the sole criterion for the evaluation of educational success.
- Mandates for the achievement of minimal competencies should not limit the school mathematics program in its broader range of essential goals.
- Legislation should not determine educational goals but when necessary should facilitate the achievement of cooperatively agreed on objectives.
- Dual respect and effective articulation must become commonplace between the civic leaders who appropriately call for educational accountability and the educational professionals who must formulate responses.

The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not

Report to National Science Board (NSB) Commission on Precollege Education in Mathematics, Science, and Technology

Executive Summary

Our charge from the NSB Commission was to identify what parts of mathematics must be considered fundamental for education in the primary and secondary schools. We concluded that the widespread availability of calculators and computers and the increasing reliance of our economy on information processing and transfer are significantly changing the ways in which mathematics is used in our society. To meet these changes we must alter the K-12 curriculum by increasing emphases on topics which are fundamental for these new modes of thought.

This report contains our recommendations on needed changes—additions, deletions, and increased or decreased emphases—in the elementary and middle school mathematics curricula and a statement of more general concerns about the secondary school mathematics curriculum.

With regard to elementary and middle school mathematics, in summary, we recommend:

- That calculators and computers be introduced into the mathematics classroom at the earliest grade practicable. Calculators and computers should be utilized to enhance the understanding of arithmetic and geometry as well as the learning of problem solving.
- That substantially *more* emphasis be placed on the development of skills in mental arithmetic, estimation, and approximation and that substantially *less* be placed on paper and pencil execution of the arithmetic operations.
- That direct experience with the collection and analysis of data be provided for in the curriculum to insure that every student becomes familiar with these important processes.

We urge widespread public discussion of the implications of the changing roles of mathematics in society, support of efforts to develop new materials for students and teachers which reflect these changes, and continued and expanded experimentation within the schools.

With regard to the secondary school curriculum, in summary, we recommend:

- That the traditional component of the secondary school curriculum be streamlined to make room for important new topics. The content, emphases, and approaches of courses in algebra, geometry, precalculus, and trigonometry need to be re-examined in light of new computer technologies.
- That discrete mathematics, statistics and probability, and computer science now be regarded as "fundamental" and that appropriate topics and techniques from these subjects be introduced into the curriculum. Computer programming should be included at least for college-bound students.

Modern computer technology clearly has vast potential for enriching and enlivening the secondary school curriculum. However, we are not now in a position to make firm recommendations. There is need for research on the effects of incorporating technology into the traditional secondary school curriculum. We urge federal support for investigations into this question, including development of experimental materials and prototypes of actual school curricula.

Although we are generally optimistic about the future role of computers, we feel we must highlight one point that worries us even though it is not directly within our charge. The disparity of access between children who have a computer at home and children who do not, threatens to widen the educational gap that already exists between different economic strata. It is urgent that programs be designed to address this problem.

We clearly recognize that the most immediate problem is not the mathematics curriculum, but the

need for more, and better qualified, mathematics teachers. One section of this report is devoted to recommendations on attracting and training prospective teachers, better utilizing the talents of inservice teachers, and retraining teachers who are inadequately prepared for teaching mathematics. We feel that the coming changes in subject matter and emphasis not only will bring a new sense of vitality to K-12 mathematics, but also will encourage teachers actively to seek and participate in programs of professional development.

The Conference Board of the Mathematical Sciences stands ready to assist efforts to develop immediate strategies for addressing the teacher shortage and to develop long-term strategies for bringing about the curricular changes envisioned in this report.

I. The NSF/CBMS Meeting

In response to suggestions made at the July 9, 1982 meeting of the NSB Commission on Precollege Education in Mathematics, Science, and Technology and, specifically, to a request made by the Educators Panel of the Commission, The Conference Board of the Mathematical Sciences (CBMS) held a special meeting to address the topic The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not. The meeting was held on September 25-26, 1982 at the headquarters of the Mathematical Association of America in Washington, D.C.

Participants in the meeting included the presidents of the American Mathematical Society, National Council of Teachers of Mathematics, Mathematical Association of America, American Mathematical Association of Two-Year Colleges, and Society for Industrial and Applied Mathematics. Two members of the Commission, Frederick Mosteller and Katherine Layton, and two members of the Commission staff, Ray Hannapel and Mary Kohlerman, also participated in the meeting. The other participants were representatives of the CBMS constituent organizations and the CBMS officers.

Six position papers on the fundamentals in the mathematics curriculum were written expressly for this conference. These position papers, along with various other background materials were distributed to all participants prior to the meeting.

The initial portion of the meeting was devoted to discussion of the position papers. Following this, participants joined working groups to address the question of what is still fundamental and what is not in K-8 and in secondary school mathematics. A general discussion of the written reports of the working groups was held during the last hour of the Saturday session.

On Sunday, new working group assignments were made to discuss the implications of changes in the K-12 mathematics curriculum. The reports from these groups were discussed in the closing session of the conference.

II. Recommendations To The Commission

Introduction

In the limited time available during the conference, it was not possible to establish full consensus on every detail of the working group reports. However, there clearly was broad consensus on the need to incorporate calculators and computers, as well as additional data analysis, into the K-12 curriculum and to make the necessary adjustments in the mathematical topics and modes of thought traditionally taught at these grade levels.

Some detailed recommendations on the fundamentals in the K-8 curriculum, what should be emphasized more and what should be emphasized less, are given in the working group report "Elementary and Middle School Mathematics." The corresponding adjustments needed in the secondary school curriculum, where the impact of technology is even greater, are described in more general terms in the two reports "Traditional Secondary School Mathematics" and "Non-traditional Secondary School Mathematics." In this area much more investigation and experimentation are required before a firm consensus can be reached.

Recommendations on dealing with the challenge of providing children with access to, and understanding of, computers and calculators pervade this entire report. They are dealt with specifically in the report "The Role of Technology." A statement of the relationship between the mathematics curriculum and what is, or can now be, taught in other disciplines is given in the report "Relations to Other Disciplines." The report entitled "Teacher Supply, Education, and Re-education" contains a variety of recommendations on attracting and retaining well-qualified mathematics teachers.

The question of the total time in the school curriculum which should be devoted to the study of the mathematical sciences was not addressed in any detail at the meeting. The general feeling was that, at the primary level, there appears to be an approximate balance between topics needing more emphasis and those needing time. At the secondary level, it is not yet clear how much time, in addition to the time that can be gained by streamlining the traditional mathematics curriculum, will be needed for discrete mathematics, probability and statistics, and computer science. This can only become clear after detailed examination of model mathematical sciences curricula and careful consideration of the many competing demands for time in the overall school curriculum.

There was general agreement at the conference that the most pressing immediate problem is the need for more, and better qualified, teachers in the classrooms. No curriculum, no matter how well-founded, can possibly succeed without dedicated and competent teachers to teach it. However, many participants felt that appropriate changes in the curriculum at this time, rather than detract from efforts to deal with the teacher shortage, could bring a new sense of vitality to K-12 mathematics and could serve to encourage teachers to actively seek and participate in programs of professional development.

Participants in the conference were also in agreement that their suggestions, even if influential in full, cannot be expected to constitute a "cure-all" for all the shortcomings of K-12 mathematics. In fact, a fundamental improvement in K-12 mathematics can be hoped for only within the framework of a general improvement of the total school environment. Remedies for the difficulties facing the teaching community (low teachers salaries, low prestige, lack of support by society, lack of discipline in the classroom, irregular attendance, etc.) are societal in nature and fall outside both the mandate and the competence of this group.

Some Additional Recommendations

In addition to the concerns and recommendations in working group reports, a few points were emphasized in the general discussions which are of vital importance in the implementation of any curricular changes:

- Textbooks

  Textbooks play a key role in the mathematical sciences curriculum at all levels. Any major changes in the curricula at the elementary, middle, or high school levels must be accompanied by corresponding changes in textbooks. For this to happen, the groups responsible for preparing textbook series and for adopting textbooks must be deeply involved in efforts to up-date these curricula.

- Testing

  To a large extent the grade and high school teachers are under strong pressure to train their pupils so as to maximize their chances of doing well on standardized tests. As long as these tests stress computations, the pupils are bound to be drilled in computations, regardless of any other guidelines the teachers may have received, and even contrary to the sounder convictions the teachers themselves may have.

  We call the attention of the Commission to the power and influence of standardized tests. Properly modified, these can have considerable effect in hastening the hoped-for improvements in the teaching of mathematics in grades K-12.

- Articulation

  The entrance requirements and course prerequisites of the nation's colleges and universities are major factors in determining the topics in the secondary school curriculum as well as the amount of time devoted to them. Efforts to change the curriculum at the secondary level must be carried out in a cooperative effort with the colleges and universities.

- Equal Access

  The disparity of access to computers between children who have a computer at home and children who do not threatens to widen the educational gap that already exists between different economic strata. It is urgent to design programs to address this problem.

- Women and Minorities

  The conference noted with satisfaction the improvement during recent years in the participation of women in upper secondary mathematics. The many efforts that have led to this improvement must continue to be supported. We look forward to corresponding success with minority and handicapped students.

Working Group Report: Elementary and Middle School Mathematics

Arithmetic and, more generally, quantitative thought and understanding continue to become more important for more people, but the importance of various aspects of arithmetic has changed and will continue to change as computers and calculators become pervasive in society. The suggestions below are designed better to equip students for life and effective functioning in the developing age of technology. We believe implementation of these suggestions into the K-8 curriculum will make students more adaptive to future change, better equipped to use modern technology, better grounded in the mathematical basis for other sciences, and better grounded for further school mathematics.

A principal theme of K-8 mathematics should be the development of number sense, including the effective use and understanding of numbers in applications as well as in other mathematical contexts.

The changes we propose are fairly substantial, but are primarily in emphasis rather than in overall content. We believe they are consistent with, and are natural outgrowths of, recommendations relative to K-8 education of the earlier valuable documents, Basic Mathematical Skills by NCSM and An Agenda for Action by NCTM.

When implemented, the changes will be only modest at the K-3 level but more significant at the 4-6 and 7-8 grade levels. They essentially replace excess drill in formal paper-and-pencil computations with various procedures to develop better number sense on the part of the student.

Here is a list of special concerns:

1) Thorough understanding of and facility with one-digit number facts are as important as ever.

2) The selective use by students of calculators and computers should be encouraged, both to help develop concepts and to do many of the tedious computations that previously had to be done using paper and pencil.

3) Informal mental arithmetic should be emphasized at all levels, first aimed at exact answers and later at approximate ones. Such activity is necessary if students are to be able to decide whether computer or calculator printouts or displays are reasonable and/or make sense. Informal mental arithmetic involves finding easy, not formal algorithmic, ways of looking at number relationships.

4) There should be heavy and continuing emphasis on estimation and approximation, not only in formal round-off procedures, but in developing a feel for numbers. Students need experience in estimating real world quantities as well as in estimating numerical quantities which appear in complicated form. Methods requiring explicit (right or wrong) answers should be used where possible to help develop estimating procedures. For example, many exercises on comparing fractions with easy ones (e.g., 12/25 with 1/2, and 103/299 with 1/3) can be used to get students to think of complicated fractions as close to, but less than (or more than), easy fractions.

5) There should be a heavy and continuing emphasis on problem-solving, including the use of calculators or computers. Trial and error methods, guessing and guestimating in solving word problems should be actively encouraged at all levels to help students understand both the problems and use of numbers. Naturally, examples and illustrations should be appropriate to the students' age, interest, and experience.

6) Elementary data analysis, statistics, and probability should be introduced, or expanded in use, including histograms, pie-charts, and scatter diagrams. The understanding and use of data analysis is becoming a vital component of modern life. The collection and analysis of data should include personal data of meaning to students, e.g., number of siblings, students' ages, heights and weights, data culled from newspapers, almanacs, and magazines, random data such as that produced by urn schemes and data from experiments in other school subjects.

7) Place value, decimals, percent, and scientific notation become more important. Intuitive under-

standing of the relative sizes of numbers that arise in the everyday world of applications becomes even more vital.

8) More emphasis on the relationship of numbers to geometry included, for example, number lines and plotting, should lead to better understanding of the concepts of arithmetic and of geometry.

9) Understanding of fractions as numbers, comparison of fractions, and conversions to decimals should have more emphasis while drill on addition, subtraction, and division of numerical fractions with large denominators should have less.

10) Drill on the arithmetic operations on three-(and larger) digit numbers should be de-emphasized. Such computations can and should be done by calculators and computers.

11) Intuitive geometric understanding and use of the mensuration formulas for standard two- and three-dimensional figures should be emphasized. More stress on why the formulas make sense is needed.

12) Function concepts including dynamic models of increasing or decreasing phenomena should be taught. (For more details, see 40 in "Traditional Secondary School Mathematics.")

13) The concepts of sets and some of the language of sets are naturally useful in various mathematical settings and should be used where appropriate. However, sets and set language are useful tools, not end goals, and it is inappropriate to start every year's program with a chapter on sets.

14) Based on motivation from arithmetic, algebraic symbolism and techniques should be encouraged, particularly in grades 7 and 8.

15) More extensive use of mathematics and computers in social science and science courses should be actively pursued. We encourage the consideration of this matter by experts in these fields and welcome opportunities to collaborate on further work in this area.

A discussion of possible computer programming for computer literacy courses is left to other groups for further study.

We call the Commission's attention to the fuller discussions and comments related to the K-8 curriculum in various position papers discussed at this conference including those of Anderson, Hilton, Pollak, and Willoughby.

Implementation Concerns

1) We hope the Commission will encourage widespread public discussion of the implications for K-8 mathematics of the changing role of arithmetic in society. As an early step, we suggest discussions and conferences between teachers, supervisors, mathematics educators, mathematicians and editors of textbook series concerning this report and others on the same general topic. Such conferences could be quite inexpensive if most participants are local.

2) We hope the Commission will seek ways to encourage the development and use of textbooks for students and of teacher-training materials in the spirit of the suggestions made above.

3) We hope the Commission will seek ways to encourage changes in standardized tests toward number sense and problem-solving and away from single-operation computational skills.

4) We hope the Commission will encourage school systems to reassign interested teachers at the 4-6 grade level to become specialists at teaching mathematics or other disciplines. One mode might be a simple trade of classes between teachers with each teacher concentrating in areas of particular interest and competence. The needed changes in subject matter emphasis will be much easier to effect if those actually teaching any subject are selected for their special interests and aptitudes. Special inservice training programs should be developed for all such semi-specialized teachers, whatever their subject.

5) We hope the Commission will seek ways to improve the status of teachers and the conditions under which teachers attempt to do the important and difficult job of educating future citizens.

6) We believe that the needed changes can be brought about somewhat gradually and with general support of those concerned. There already is discussion in teacher and supervisor groups concerning many of the ideas put forth here. The proposed changes generally involve modifications in the way mathematics is introduced and used in schools rather than adding new subject matter. The changes should permeate texts and not just be add-ons that can be ignored. There appears to be an approximate balance in time between topics needing more emphasis and those needing less. With the exception of computer use and the possible exception

of parts of data analysis, the topics needing added emphasis have been taught and learned in American schools at various times and places in the past. The diminished role of paper and pencil computation is perhaps the topic which will provoke most concern and possible disagreement.

Working Group Report: Traditional Secondary School Mathematics

Current secondary school mathematics curricula are organized into separate year-long courses covering algebra, geometry, and precalculus topics. There are proposals that challenge this traditional division of school mathematics and the position of calculus as the primary goal for able college-bound students. Thus, the following analysis used conventional course headings for discussion of proposed changes in traditional topics, not as endorsement of the status quo.

1) **Overall Recommendation:** The traditional component in the secondary curriculum can be streamlined, leaving room for important new topics. However, since breakthroughs in technology which allow this streamlining are so recent and the conceivable implications so revolutionary, it is not yet entirely clear what specific changes are appropriate.

2) **Algebra:** The basic thrust in Algebra I and II has been to give students moderate technical facility. When given a problem situation, they should recognize what basic algebraic forms they have and know how to transform them into other forms which might yield information. In the future, students (and adults) may not have to do much algebraic manipulation—software like mu-Math will do it for them—but they will still need to recognize which forms they have and which they want. They will also need to understand something about why algebraic manipulation works, the logic behind it. In the past, such recognition skills and conceptual understanding have been learned as a by-product of manipulative drill, if learned at all. The challenge now is to teach these skills and understanding even better while using the power of machines to avoid large time allotments to tedious drill. Some blocks of traditional drill can surely be curtailed, e.g., numerical calculations using look-up and interpolation from logarithm and trigonometry tables.

3) **Geometry:** A primary goal of the traditional Euclidean geometry course is to develop logical thinking abilities. But not every fact need be given a rigorous proof to pursue this goal. Nor need this be the only goal of geometry, nor geometry the only means towards this goal.

We recommend that classes work through short sequences of rigorously-developed material, playing down column proofs, which mathematicians do not use. These proof sequences should be preceded by some study of logic itself. Important theorems not proved can still be explained and given plausibility arguments, and problems involving them can be assigned. The time which becomes available because proofs are de-emphasized can be devoted to study of algebraic methods in geometry, analytic geometry and vector algebra, especially in three dimensions. Work in three dimensions is essential if one is to develop any pictorial sense of relations between many variables, and handling many variables is essential if one is to model phenomena realistically.

There is much room for using computers in geometry. The power of graphics packages makes it much easier for students to get a visual sense of geometric concepts and transformations. The need to use algebraic descriptions of geometric objects when writing graphics programs reinforces analytic geometry. Finally, the algorithmic thinking needed to write programs bears much resemblance to the thinking required to devise proofs.

4) **Precalculus:** What often happens in this course is that students see the same topics yet another time, with more drill but with little new perspective. For better students there may not be a need for a precalculus course if drill is no longer so important and if algebra and geometry are done "right," with the concepts made clear. For instance, one justification for the precalculus course is the perceived need to develop the idea of functions; functions appear in Algebra I and earlier, but current teaching may give too static an understanding. With computers, the concept of function can be made central earlier and more clearly. The computer supports qualitative analysis of the graphs of functions, in a dynamic mode of display, and also allows detailed analysis of zeros, rates of change, maxima/minima, etc.

5) Algorithmics: Computers and programming have made the creative human talents and skills involved in developing and analyzing algorithms extremely important. These talents and skills, emphasized by the group on non-traditional topics, can be exercised quite naturally through traditional topics as well. Much of high school algebra consists of systematic methods for handling certain problems, e.g., factoring polynomials. Such methods are algorithms. Instead of making the student carry out such methods with paper and pencil a boring number of times, have the student do it just a few times and then program a computer to do it. The understanding gained should be at least as great.

6) The Average Student: For the many students in secondary school who are not specially talented in mathematics and not headed for careers in science or technology, current programs are a source of discouragement, anxiety and repetition in a dull "basic skills" program which serves them poorly. We cannot ignore the needs of this large and important group. Computers, as mathematical tools and media of instruction, offer a fresh window into mathematics for them.

7) Cautions: We have suggested that technology provides an opportunity to devote less time to traditional techniques while boosting understanding and allowing more time for more complex, realistic problem solving. However, there are several cautions. First there are widespread and deep reservations about how much traditional goals should give way to technology. Second, there is little research data on the feasibility of such changes, and there are almost no prototype school curricula embodying the new priorities. Experimental programs, and research on the results, must be given major support. Third, changes in secondary programs must be carefully articulated with the expectations of colleges and employers, who often have conservative views about curricula. Finally, the syllabi of an extensive range of standardized tests play a very influential role in setting curricula and the actual classroom emphases of teachers. If curricula are to change, the tests must be changed. Clearly, strong national leadership and cooperation are necessary, from teachers, mathematicians and public policymakers, to meet these challenges and implement significant change.

Working Group Report: Non-Traditional Secondary School Mathematics

On two basic principles the panel was unanimous

- There is need for substantial change in both the subject matter of and the approach to teaching in secondary school mathematics.
- If changes are to be made in secondary school mathematics, we must make haste slowly, taking care at all times to insure full consultation with and support from the secondary school mathematics teaching community.

Our specific recommendations are grouped under five headings: Subject Matter, Approach to Teaching, The Use of New Technology, Teacher Training, and Implementation.

1) Subject Matter: Careful study is needed of what is and what is not fundamental in the current curriculum. Our belief is that a number of topics should be introduced into the secondary school curriculum and that all of these are more important than, say, what is now taught in trigonometry beyond the definition of the trigonometric functions themselves. These topics include discrete mathematics (e.g., basic combinatorics, graph theory and discrete probability), elementary statistics (e.g., data analysis, interpretation of tables, graphs, surveys, sampling) and computer science (e.g., programming, introduction to algorithms, iteration).

2) Approach to Subject Matter: The development of computer science as well as computer technology suggests new approaches to the teaching of all mathematics in which emphasis should be on

- algorithmic thinking as an essential part of problem solving
- student data gathering and investigation of mathematical ideas in order to facilitate learning mathematics by discovery.

3) Technology: New computer technology allows not only the introduction of pertinent new material into the curriculum and new ways to teach traditional mathematics but it also casts doubt on the importance of some of the traditional curriculum, just as the hand calculator casts similar doubts about instruction in arithmetic. Particularly noteworthy in this context at the secondary level are

- Symbolic manipulation systems which even now, but certainly far more in the near future, will allow students to do symbolic algebra (and calculus) at a far more sophisticated level than they can be expected to do with pencil and paper.

- Computer graphics and the coming videodisc systems which will enable the presentation and manipulation of geometric and numerical objects in ways which should be usable to enhance the presentation of much secondary school mathematical material.

One caveat which we would stress is that this technology and related software packages must be used not to enable students to avoid understanding of the essential mathematics but rather to enhance such understanding and to allow creative experimentation and discovery by students as well as to reduce the need for tedious computation and manipulation.

4) **Teacher Training:** There are two aspects of this

a) Retraining of current teachers in the new topics, approaches and technology. One possible new approach to this might be the use of college students to aid and instruct secondary school personnel as part-time employees and perhaps through such incentives as forgiveness of student loans.

b) Education of new teachers. Crucial to long-term solution of the secondary school mathematics education problem is that the requirements for degrees in mathematics education be, as necessary, changed to incorporate modern content and approaches. In particular, we believe that all prospective teachers of secondary school mathematics should be required to take at least

- one year of discrete mathematics in addition to traditional calculus requirements
- one semester or one year of statistics (with focus on statistical methods rather than mathematical statistics)
- one year of computer science.

5) **Implementation:** We recognize that the kinds of changes proposed here not only require much more study than has been possible by our panel but that also they will never be implemented unless there is dedicated cooperation among:

- secondary school teachers of mathematics and their professional organizations
- college curriculum people in schools of education and in mathematics departments and including their organizations
- state and local education authorities and their organizations.

A conference at an early date bringing together these groups to discuss the relevant problems and plan future action might be the most fruitful next step to provide some momentum for the changes we believe are necessary.

Working Group Report: The Role of Technology

Computers and related electronic technology are now fundamental features of all learning and working environments. Students should be exposed to and utilize this technology in all aspects of school experience where these devices can play a significant role.

We recommend:

1) The potential of technology for enhancing the teaching of mathematics and many other subjects is vast. Development of such resources should be supported at a national level. Specific examples include computer-generated graphics, simulations, and video-disc courseware materials. There should be efforts to create a network providing easy access to such banks of material.

2) While computing technology offers promise to enhance learning, differential access to the benefits of that technology could widen the gaps in educational opportunity which already separate groups in our society. It is imperative that every effort be made to provide access to computers and their educational potential for all sectors of society.

3) As a general principle, each mathematics classroom should have available computers and other related electronic technological devices to facilitate the computing and instruction required for mathematics learning and competency. Such availability of computers and other electronic technological devices in the mathematics classroom is as important as the availability of laboratory equipment for science instruction.

4) Hand calculators should be available in mathematics classrooms (both in elementary and secondary schools) for students on the same basis that textbooks are now provided.

5) Support should be given for broad developments in software that may be useful in the schools. School districts should encourage their teachers and students to engage in cooperative development activities and to find ways to recognize and disseminate the products of those efforts.

6) Computer literacy involves not only the use of computers to accomplish a great spectrum of tasks but also a general understanding of the capabilities and limitations of computers and their significance for the structure of our society. Development and implementation of appropriate programs to teach these more general concepts should be supported.

7) Possible curricular changes emanating from technological changes will require careful study and deliberation over a long period of time. This activity must be encouraged and supported from a national level. The exploratory projects should bring together teachers, curriculum developers, mathematicians, and affected interested parties from business and industry. The new programs developed should be tested extensively in a variety of settings to insure that they work with real students and schools before extensive implementation is attempted.

8) The interplay between word-processing, computers, data bases, and data analysis methods assist in breaking down barriers between disciplines thus offering an opportunity for schools to provide a range of holistic problem-solving experiences not typical in school today. Using the technology as an aid, students can plan and conduct data collection, analysis, and report writing that is realistic, attractive, and far beyond normal expectations in today's schools.

9) The need for well-trained, highly qualified teachers of mathematics is a must in a technological society. Support should be given to organizing programs for inservice training and retraining of current teachers of mathematics (elementary and secondary) who are inadequately prepared to teach a technologically-oriented curriculum, but have the capacity to profit from such programs to strengthen their mathematical preparation and teaching skills.

10) While technology provides opportunities, it also makes demands. The world becomes a more complex place in which to live. If we are to insure that a broad spectrum of society can function and participate actively in the business/industrial community and decision making of the country, it is imperative that students become adept in the precise, systematic, logical thinking that mathematics requires.

Working Group Report: Relations to Other Disciplines

Along with the effects of computational technology on the mathematics curriculum, it is also necessary to consider how this technology and the proposed curricular changes affect the relationships between the curricula in other disciplines and the curriculum in mathematics. We have interpreted the phrase "other disciplines" rather broadly.

First, using a narrow view, we must look at the effects these curricular changes will have on science education. There has always been a necessary interaction and coordination between the science and mathematics curricula, particularly with the physical sciences. At a minimum, this revised curriculum, which encourages student use of calculators and computers and emphasizes a good sense of estimation, provides an opportunity for elementary and high school education to be more realistic and eliminate the use of specialized problems with "easy numbers." If we raise our sights a bit, there is an opportunity for a better coordinated and integrated total science education. Furthermore, the introduction of statistical ideas, data handling procedures, and discrete mathematics provide an opportunity for a more mathematical discussion of social science problems at the elementary and high school levels. Similarly, changes in currently available tools will undoubtedly affect courses in business and commercial programs.

Related questions arise on the other side. What do the school programs and the college programs in natural sciences, social science, and business require in the mathematical preparation of entering students? We believe the suggested curriculum can only be an improvement, but discussion with leaders of those disciplines are required.

Taking the broad view, we also believe that this modified curriculum, which provides students with the same (or greater) ability to use mathematics as well as an ability to use and appreciate the technology, will provide for a wiser citizenry. The graduates of such a program should be better equipped to deal with "poll results" and statistical data references to the economy and sociological problems.

We believe there is one serious area in which the nation needs more data for the development of an appropriate mathematics curriculum. Namely, what are the needs, in terms of mathematical skills, of the students who seek technical vocational employment without going on to further schooling? Furthermore, what are the mathematical needs of students going on to technical or vocational schools? Although we do not know the answers, we believe the new curriculum will do at least as good a job as the existing one. A conference or meeting to explore this area would be an excellent idea and would complement our work.

Working Group Report: Teacher Supply, Education, and Re-Education

Efforts to improve and up-date the mathematics curriculum and to increase the mathematics, science, and technology literacy of all citizens require the support of qualified mathematics teachers at all levels. At present there is a serious and well-documented shortage of qualified teachers of mathematics at the elementary and secondary school levels in many areas of the country. Economic, employment, and social conditions forecast that the current short supply may indeed be a long-term problem. Furthermore, even in geographic locations where adequate supplies exist, the frequent turnover of mathematics teachers tends to impede learning.

The following recommendations address the need to increase the supply of mathematics teachers as well as to improve the quality of the teacher and thereby, the learning, of mathematics:

1) While state and local efforts by industry, business, and academe to deal with the teacher shortage are laudable and should continue, the magnitude of the problem is national in scope. An articulated national commitment with federal leadership and support is needed for its resolution. The public should be made aware of the problem through more effective publicity.

2) Incentives of all types need to be studied to attract and retain qualified teachers of mathematics. Financial incentives should be given special attention with priority assigned to those which do not create undue inequities and tensions among colleagues, in order to avoid being counterproductive.

Examples of possible incentives and support systems include the following:

a) Forgiveness of student loans and/or interest on loans for those who enter the teaching field.

b) High entry level salaries for special expertise (e.g., computer training).

c) Reduced teaching loads to allow teachers to pursue graduate study or other advanced training in the mathematical sciences and applied areas.

d) Financial support of graduate study or other advanced training in the mathematical sciences and applied areas.

e) Salary differentials by discipline.

f) Summer positions and other cooperative arrangements with business and industry to supplement a teacher's income (with the obvious caveat that the short supply of teachers is largely due to the fact that higher industrial salaries lure teachers away; industry would have to be discouraged from using this arrangement for recruitment purposes.)

3) In an era when content and technology are changing so rapidly, incentives are needed to keep qualified teachers in the field abreast of current trends in the mathematical sciences. Inservice workshops, NSF-type institutes, retraining courses, industrial experiences, and other forms of continuing education can serve to refresh the faculty and renew their commitment to teaching.

4) In some parts of the country, teachers from other disciplines are being assigned to teach mathematics classes. These teachers need special subject matter training and assistance in developing appropriate teaching strategies in order to succeed in their new assignments.

5) Encouraging colleges and universities to loan their faculty, and business and industry to loan their mathematically-oriented employees to teach courses in the secondary schools could be mutually beneficial. Qualified retirees or near-retirees might also be recruited to enter the teaching field. (Of course, the issues of appropriate teacher training and certification need to be addressed.)

6) In states where this is not the norm, it is recommended that teacher certification requirements

be stated in terms of the specific topics to be covered in the subject area rather than in terms of just total number of credits.

7) Recommendations regarding the mathematical fundamentals to be covered in educating qualified teachers of mathematics include:

a) Elementary Level

It is strongly suggested that mathematics at the elementary school level be taught by teachers who specialize in mathematics. Whether the teacher specializing in mathematics should be assigned to all grades or just to grades 4-6 (or 4-8) requires further study. An alternative approach would be to identify those teachers in a given school who most enjoy teaching mathematics. Those teachers could be assigned to teach all mathematics courses across a grade level, while other teachers do similarly in reading and writing.

The following recommendations pertain to both the regular elementary school teacher and the teacher specializing in mathematics:

For entry into the mathematics education program for elementary school teachers, at least three years of college-track mathematics in high school is recommended. College mathematics courses should provide a sufficient background to understand the relationships between algebra and geometry, functions, elementary probability and statistics, instruction in the use of the handheld calculator, and some exposure to computers. Creative approaches to problem solving should also be included in the curriculum. Training should be at least one level above what is being taught. This background is particularly important in light of children's awareness of the world around them through television, other media, computers, and so on.

b) Secondary Level

Secondary school mathematics teachers should have course work in mathematics equivalent to a major in mathematics. Requirements for those who will teach mathematics should include the equivalent of a two-year calculus and linear algebra sequence, discrete mathematics, probability and statistics, and appropriate computer training. These courses should develop in the student a sense of "mathematical maturity" in the approach to problem solving.

Conclusion

The recommendations cited here require careful planning and implementation. With high technology a mainstay of our present and future society, it is imperative that we recognize and promote mathematics as a powerful, useful, and enjoyable component of our lives.

Improving What Is Taught and Learned*

The National Science Board Commission on Precollege Education in Mathematics, Science and Technology

In addition to improving the quality of mathematics, science and technology teaching, efforts must be made to improve what and how students learn about those subjects. Immediate and significant progress can be made by increasing the amount of time students spend learning mathematics, science and technology. Persistent improvements, however, require redesigning educational objectives and related curricula and instruction to conform to those new educational objectives.

A. *Raising Requirements for Mathematics, Science and Technology Education*

This Commission is convinced that increased time-on-task is a crucial element to improving mathematics, science and technology education in our Nation. Short-term improvements will result from increasing the amount of time spent studying these subjects in the elementary school and by increasing the requirements for high school graduation and college entrance.

1. Raising Requirements in K-8

Mathematics, science and technology education must begin at an early age. Knowledge and skills in specific subject areas must be developed in incremental steps, and equally important, good habits of study—the ability to apply oneself to more difficult tasks—must be nurtured and reinforced over time.

Therefore, the Commission recommends:

- Local school districts should revise their elementary school schedules to provide more time-on-task for the study of mathematics, science and technology. In grades K-6[1], a minimal daily allocation of 60 minutes per day for mathematics and 30 minutes for science should be required.
- A full year of mathematics and of science and technology should be required each year in grades 7 and 8 (high school requirements are discussed in the next section).

2. Raising Requirements for High School Graduation and College Entrance

The trend in the 1960s toward reduced high school graduation requirements in many school systems and reduced admissions criteria in many college systems is frequently blamed for producing a decreased emphasis on academic standards and achievement in secondary schools. In fact, many students graduate from high school with only one year of mathematics and one year of science. Lessening rigor in secondary schools directly affects priorities in elementary schools. The Commission believes that high school graduation and college entrance requirements in mathematics and science, which in many school systems now amount to only a bare minimum, should be significantly increased and this should be done forthwith. This is already occurring in a number of school and university settings.

The Commission is aware that education must meet the needs of all students, and that not all will pursue similar careers. To recommend equally demanding courses of study in mathematics, the

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**From* Educating Americans for the 21st Century: A plan of action for improving mathematics, science and technology education for all American elementary and secondary students so that their achievement is the best in the world by 1995. *The National Science Board Commission on Precollege Education in Mathematics, Science and Technology, 1983.*

[1] *When kindergarten is taught for only one-half day, these requirements would, of course, be halved for kindergarten.*

sciences and technology for each secondary school student may not be sound. This is not to say that high expectations should not drive all of our educational endeavors. But, rather, reality suggests that we prepare the academically talented for further study in our colleges and universities in a rigorous manner and perhaps in areas different from those who will not choose to continue formal education in institutions of higher learning.

Thus, the Commission further recommends:

- Each state should institute statewide performance standards and strict high school graduation requirements, and each local school district should institute more stringent grade promotion requisites.
- All secondary school students should be required to take at least three years of high school mathematics including one year of algebra (see Exhibit B for suggested course content), and at least three years of science and technology, including one semester of computer science, prior to high school graduation. This requirement should be mandated at the state and/or local level to take effect no later than September 1, 1985[2].
- All schools should provide opportunities for their students to develop their mathematical and scientific skills to the limits of their abilities and should offer appropriate sequences of courses for students at various levels of ability.
- Steps should be taken to phase in higher mathematics and science entrance requirements for all colleges and universities (a development now underway in many institutions of higher education). Such college requirements should include four years of high school science, including physics, chemistry and one semester of computer science, four years of mathematics, including a second year of algebra and coursework covering probability and statistics.
- Every school district and each middle and secondary school should establish mechanisms and procedures which obligate specific school personnel to inform, in time for such courses to be taken, all students of these college entrance requirements.
- Guidance counselors play an important role in academic and career advising of students. School districts must give special consideration to the preparation and responsibilities of those persons who occupy such positions. In addition to competence in the technical areas of advising and counseling, guidance counselors should be selected and trained with appropriate emphasis on their sensitivity and understanding of the nation's commitment to abolishing discrimination based on race, sex, ethnicity or socioeconomic circumstances.
- School districts and community colleges should cooperate in providing appropriate opportunities for students whose mathematics and/or science preparation is inadequate to allow them to take the next steps in their education.

B. New Educational Objectives

Throughout this Report, the Commission has asserted that American students require a substantially improved background in mathematics, science and technology. Many of the courses now taught in most American schools badly need revision and updating; better trained teachers must have more up-to-date courseware in order to teach effectively. These needs result from the explosion of knowledge in scientific and mathematical fields, the availability of new technologies for the communication of this knowledge, the recent results of cognitive and behavioral research about how students think and learn, and the failure of some teachers to keep their skills current. Some improvements can be made in the short term by increasing high school graduation and college entrance requirements (see discussion above). For the long term, however, a continuing process of planning and review must be established at all levels of America's educational systems. In the process, new educational objectives will have to be set and met. With this capacity, our educational system may be able to avoid periodic crises.

The Commission sets forth herein a suggested set of educational objectives. This should not be construed as a suggestion for the establishment of a national curriculum; rather these are guides that state and local officials might use in developing curricula for local use.

Factors that must now be considered in setting new educational objectives are:

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[2] *Requirements should be instituted forthwith, but the Commission recognizes these requirements must be phased in so as to avoid penalizing current students who do not have sufficient time remaining before graduation to meet all of these new requirements; those students should have to fulfill as many of these requirements as possible.*

- The need to expand the focus of mathematics, science and technology education from only the pre-professional to all students;
- The changes in what students should know about mathematics, science and technology;
- The availability of new communication and educational technologies;
- The promise for education resulting from recent work in the social and behavioral sciences;
- The tendency of the present system to motivate only the already motivated and pre-professional students to continue to study and enjoy the fields of mathematics and the sciences;
- The evidence that students entering primary school exhibit a natural curiosity about numbers and about the world around them, but are discouraged from consideration of serious study in these fields early in the elementary grades due to inadequate teaching or lack of motivation.[3]

In the Commission's discussions with a variety of groups, including the education, business and government communities, very few endorsed an emphasis on specific job-related skills. Rather, there was general agreement that our contemporary society requires youth who are "trainable," that is, who have the capability of continuing to learn as changes take place in our society, whether at work or elsewhere. Teaching specific job-related skills instead of providing a strong general education is deemed ill-advised because specific job skills might be needed only for a short span of time.

Suggestions for selection criteria, course topics and student outcomes are presented in Exhibit B and summarized below. Many of these suggestions were developed by conferences sponsored by the Commission. We are confident that these suggestions will move the Nation's educational systems in appropriate directions. They are presented to stimulate further discussion and debate. The Commission urges their careful review by organizations of scientists, mathematicians, teachers, technologists and educational researchers.

1. Objectives for Mathematics Education

Mathematics is a way of thinking that opens doors to new knowledge in virtually every field (e.g., art, music, social sciences) and is essential for understanding the sciences. The applications of various aspects of mathematics in the sciences and engineering are changing and will continue to change as computers and calculators become ever more pervasive. The skills listed below provide a base for the study of other disciplines and are considered essential for the use of current technology and for further study in the mathematical sciences.

Analysis of current student performance in mathematics—particularly the use of mathematical skills in unfamiliar areas—indicates that they are learning to be technicians but not problem solvers. Opportunities should be provided for the application of arithmetic and mathematics in a variety of areas—in the natural and social sciences, in consumer-related experiences and in other real-life situations where analysis through mathematics is possible. In this context, computers and calculators make it possible to present simulation as a problem-solving tool with important applications in virtually every aspect of elementary and secondary learning.

Mathematics instruction at the *elementary level* should be designed to produce the following outcomes:

- Comprehensive understanding of and facility with one-digit number facts, place values, decimals, percentages and exponential notations;
- Skill in informal mental arithmetic, estimation and approximation;
- Ability to use calculators and computers selectively;
- Basic understanding of elementary data analysis, simple statistics and probability, and fractions;
- Ability to use some algebraic symbolism and techniques;
- Thorough understanding of arithmetic operations and knowledge of when each should be used.

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[3] *Paul DeHart Hurd,* State of Precollege Education in Mathematics and Science, *paper prepared for the National Convocation on Precollege Education in Mathematics and Science, Washington, D.C., May 12-13 1982, sponsored by the National Academy of Sciences and the National Academy of Engineering, Washington, D.C.*

At the *secondary level* there is a need to examine the content, emphasis, and approaches of courses in algebra, geometry, precalculus methods and trigonometry. Some components in the traditional secondary school mathematics curriculum have little importance in the light of new technologies. The current sequence which isolates geometry in a year-long course, rather than integrating aspects of geometry over several years with other mathematics courses, must be seriously challenged. Some concepts of geometry are needed by all students. Other components can be streamlined, leaving room for important new topics.

- — Discrete mathematics, elementary statistics and probability should now be considered fundamental for all high school students.

The development of computer science as well as computer technology suggests new approaches to the teaching of all mathematics in which emphasis should be on:

- — algorithmic thinking as an essential part of problem solving;
- — student data-gathering and exploration of mathematical ideas in order to facilitate learning mathematics by discovery.

New computer technology allows not only the introduction of pertinent new material into the curriculum and new ways to teach traditional mathematics, but it also casts doubt on the importance of some of the traditional curriculum. Particularly noteworthy in this context at the secondary level are:

- — Symbolic manipulation systems which even now, but certainly far more in the near future, will allow students to do symbolic algebra at a far more sophisticated level than they can be expected to do with pencil and paper;
- — Computer graphics and the coming interactive videodisc systems which will enable the presentation and manipulation of geometric and numerical objects in ways which should be usable to enhance the presentation of much secondary school mathematical material.

We stress that the use of this technology and related software packages is not a substitute for the understanding of the essential elements of mathematics, but rather is a means to enhance understanding and stimulate creativity.

2. Objectives for Science and Technology Education

Science and technology are integral parts of today's world. Technology, which grows out of scientific discovery, has changed and will continue to change our society. Utilization of science in the solution of practical problems has resulted in complex social issues that must be intelligently addressed by all citizens. Students must be prepared to understand technological innovation, the productivity of technology, the impact of the products of technology on the quality of life, and the need for critical evaluation of societal matters involving the consequences of technology.

Further, the nature of scientific inquiry and observation presents frequent opportunities for experiencing success with original ideas. Such inquiry does not require unique answers. Students can rightly and successfully report what they have seen and found. This type of experience should be encouraged.

Businesses will require, to an increasing degree, people who are knowledgeable about science and technology. Science and technology instruction at the elementary and secondary levels should be designed to produce the following outcomes:

- — Ability to formulate questions about nature and seek answers from observation and interpretation of natural phenomena;
- — Development of students' capacities for problem solving and critical thinking in all areas of learning;
- — Development of particular talents for innovative and creative thinking;
- — Awareness of the nature and scope of a wide variety of science- and technology-related careers open to students of varying aptitudes and interests;
- — The basic academic knowledge necessary for advanced study by students who are likely to pursue science professionally;
- — Scientific and technical knowledge needed to fulfill civic responsibilities, improve the student's own health and life and ability to cope with an increasingly technological world;
- — Means for judging the worth of articles presenting scientific conclusions.

The materials for broadening science instruction to achieve these outcomes must be developed and tests must be devised that will effectively measure the degree to which these educational objectives are reached. It will take time, talent, and funds to develop these materials and tests. These will not be forthcoming, however, unless top priority is given to effecting these changes and providing teachers and students with the needed new instructional experiences.

In summary, students who have progressed through the nation's school systems should be able to use both the knowledge and products of science, mathematics and technology in their thinking, their lives and their work. They should be able to make informed choices regarding their own health and life-styles based on evidence and reasonable personal preferences, after taking into consideration short- and long-term risks and benefits of different decisions. They should also be prepared to make similarly informed choices in the social and political arenas.

C. *Revamping the Curriculum*

The Commission believes the schools will have to develop for all students new, coherent patterns of K-12 mathematics, science and technology courses that are explicitly designed to meet the new educational goals set forth in the previous section. New science curricula that incorporate appropriate scientific and technological knowledge and are oriented toward practical issues are needed. They also will provide an excellent way of fostering traditional basic skills. The introduction of practical problems which require the collection of data, the communication of results and ideas and the formulation and testing of solutions or improvements would: (1) improve the use and understanding of calculation and mathematical analysis; (2) sharpen the student's ability to communicate verbally and to write precisely; (3) develop problem-solving skills; (4) impart scientific concepts and facts that can be related to practical applications; (5) develop a respect for science and technology and more generally for quantitative observation and thinking; and (6) stimulate an interest in many to enter scientific, engineering and technical careers.

Curriculum content requires revision, not only to make the substance more meaningful to students and excite them, but also to incorporate new knowledge. For example, there is now a glaring absence of technology education in American schools—a lack of curricula, course materials, demonstrated teaching methods and qualified teachers. The greater the degree to which all the sciences and technology can be integrated in new curricular approaches, the broader understanding in these fields will be.

The Commission firmly believes that thorough review of curricula should not be a reaction to periodic crises. Rather, the Commission believes that a process of continuous curricular review and updating (kindergarten through 12th grade) should be established. It should incorporate developments in various disciplines, technology and the latest understanding of the learning process.

Curriculum selection and presentation are primarily a state and local responsibility—and often are very successful—but it is generally agreed that optimum results usually require the coordinated efforts of experts from many disciplines and fields. This effort is often difficult to mount at the local or even state level. Therefore, a process which includes national leadership must be established to evaluate present courses of study, propose new approaches and promote their implementation in the classrooms of the nation.

- The National Science Foundation, which has recognized expertise in leading curriculum development, should again take the leadership role in promoting curriculum evaluation and development for mathematics, science and technology. The National Science Foundation should set up a process to evaluate existing curricula, identify good curricula, disseminate information, act as a clearinghouse and promote the development of guidelines for new curricula as necessary.
- The professional societies in mathematics, science, engineering and other technologies should play an active role in curriculum development, review and revision.

There are numerous ways in which the National Science Foundation might approach its part of this task. The Commission offers the following for serious consideration as one possible mechanism:

—A Mathematics, Science and Technology Curriculum Council might be established in the

National Science Foundation. Within the next 12 months this Council would begin a continuing process to maintain a broad overview of instructional material for elementary and secondary education. To accomplish this, it would appoint and coordinate the activities of committees to address:

Elementary Mathematics
Secondary Mathematics
Elementary Science
Secondary Science and Technology

The responsibility for each committee would include:

- Recommendations regarding course content for its subject area;
- Recommendations regarding instructional design and methodologies, including software and new hands-on approaches;
- Critical review of the available texts and other teaching and resource materials;
- Publication of these recommendations and evaluations;
- Identification of the areas where improved and/or new course materials are needed and the exploration of new methodologies;
- Identification of the areas where future research is needed in the design of curricula and the processes of teaching and learning.

Membership on these committees would consist of experts in the subject matter, including classroom teachers, representatives of professional societies and school boards, technical experts from business and government and academic educational researchers and instructors. The Commission recognizes that some of these groups have already mounted efforts in curriculum development. They should be encouraged and given the opportunity for official consultation to facilitate implementation of new courses of study.[4]

As the work of the committees proceeds and areas for development and implementation are identified, the Curriculum Council would add members representing publishers and associations of higher education, as necessary, to encourage the development and transfer of these ideas to actual material for the classroom. The Council and committees might work with demonstration centers widely dispersed around the country to evaluate new approaches. In areas where magnet schools or other exemplary schools or programs have been established, these might serve as the test centers. These test centers could play a key role in the dissemination of information on good curricula and should serve as sites for the training of teachers to use these curricula.

Content development efforts must include participation of practicing scientists, engineers, and teachers, who can be organized through their professional organizations, particularly those that have teachers and other academicians among their active members. These efforts should make use of educational technology and be closely correlated with software developers to find means to interest and excite students in mathematics, science and technology at the earliest grade possible. In considering appropriate curricula, the Commission strongly suggests that particular attention be paid to the new information and communication technologies which are revolutionizing our society. It is critical that all students gain an understanding of the operation and capabilities of computers, since this technology has become an integral part of our society and is becoming more important as time passes. States should require teachers to have computer skills. Students should have significant exposure to computers by the fourth grade and should have at least one semester of training in the use of computers by the tenth grade. Computer programming will become a tool for learning mathematics, science and technology, for learning autonomy and for learning the spirit of intellectual play. Thus, the Commission recommends:

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[4] *As a bridge between professional mathematicians, scientists and teachers, the Nation's professional societies occupy a unique position. They should play an important role in any effort to improve our educational system in mathematics, science and technology. This Commission has encountered many outstanding programs which are currently being operated by professional societies to help teachers and to motivate students. For example, the programs of the American Association for the Advancement of Science, the American Association for Engineering Education, the American Association of Physics Teachers, the American Chemical Society, the American Society for Engineering Education, the Institute for Electrical and Electronics Engineers, the Mathematical Association of America, the National Association of Biology Teachers, the National Council of Teachers of Mathematics, the National Science Teachers Association and the National Society of Professional Engineers are all noteworthy. A summary of programs provided by the professional societies is presented in Exhibit A.*

- States should take steps to ensure that all teachers and all students have access to such training.
- School districts should develop explicit plans to achieve computer literacy, since these goals must ultimately be implemented at the local level.
- Business should be invited to take part in cooperative programs which involve resource sharing and contributions of equipment.
- Individuals with expertise from business, government and the higher education community should take an active role in teaching both students and teachers about computers.

In addition, the Commission has found that curricula for separate technology courses are virtually nonexistent, and material about modern technology is generally poorly integrated into the science curriculum. Recent studies reported to the Commission have placed particular emphasis on the need for materials dealing with technology at grades 8 and 9. The Commission, therefore, recommends:

- The National Science Foundation should support the development of courses to meet this need, utilizing expertise from business both in determining course content and in formulating plans for using local technologists to assist in presenting the material.

Finally, in developing revised curricula, educators should strive to incorporate the latest information on the teaching and learning processes. The rapid growth of psychological knowledge holds some promise for improving instructional methods. Indeed, collaborative research between cognitive psychologists and researchers in mathematics and science education should be encouraged to test new concepts under classroom conditions. Consequently, the Commission recommends:

- Research into the processes of teaching and learning should be supported with Federal funds at both the basic level and the level of classroom application. This research should further the recent progress in the cognitive sciences, and particular research projects should investigate the integration of educational technologies into the processes of teaching and learning.

School Mathematics: Options for the 1990s*

Thomas A. Romberg

Chairman's Preface

In December 1983, over 40 persons attended a special working conference in Madison, Wisconsin on school mathematics. The conference was jointly sponsored by the Office of Educational Research and Improvement of the U.S. Department of Education, the National Council of Teachers of Mathematics, and the Wisconsin Center for Education Research. The participants included mathematicians, mathematics educators, psychologists, computer scientists, state and school mathematics coordinators, mathematics teachers, and publishers of educational texts, tests, and computer software.

The conference provided an opportunity for participants to review the evidence and recommendations favoring reform of school mathematics that have been made by many commissions and groups during the past two years. Based on this review, conference participants were to propose next steps that could be taken to advance the improvement process.

The conference participants very quickly found consensus on many diverse issues: calculators and computers should be used at all educational levels where appropriate; too much time is spent on elementary arithmetic skills instruction; statistics is of fundamental importance for all students and is too often ignored in the schools; working conditions in many schools do not support excellent teaching of mathematics. The problem was to bring coherence to the many claims and suggestions being made. Most of the participants worked diligently in both large group sessions and small working groups to draft statements summarizing and organizing various issues and options.

Chairman's Summary

A requirement of the conference participants was to consider evidence of the need to improve school mathematics, to develop recommendations about the nature of the needed changes, and to prepare strategies or next steps for accomplishing these changes. Participants studied the recommendations made during the past two years by the many commissions and groups critical of current practices and provided additional evidence and recommendations of their own.

Serious problems exist with respect to the opportunity most students have to learn the mathematical concepts and skills they need for college, for future employment, and for responsible citizenship. The consensus of the participants was that more than marginal improvement is needed. Sharp improvement in the nature of reform is needed, and the mathematics education community should play its part in defining the shape of school mathematics for the 1990s. The primary concerns addressed at the meeting were that most students need to learn more, and often different, mathematical topics in school and that the teaching of mathematics can be and must be significantly improved.

To provide all students with an opportunity to learn the mathematics they will need, the topics and emphases of current courses should be changed, new courses should be developed, and students should be required to enroll in more courses. To assist schools in making these changes, new curriculum materials and instructional strategies must be developed. In particular, these materials should require the use of calculators and computers when appropriate, reduce the amount of time spent on elementary arithmetic skills instruction, and add many new topics now deemed important in data collection and summarization, in statistics, and in discrete mathematics. Three recommendations for action designed to assist

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*From Thomas A. Romberg, School Mathematics: Options for the 1990s, Chairman's Report of A Conference, *U.S. Department of Education, June 1984*.

in the development of critically needed new materials are:

Recommendation 1: A task force should be organized to propose guidelines for a K-8 mathematics curriculum.

Recommendation 2: A task force should be organized to propose guidelines for a 7-14 mathematics curriculum.

Recommendation 3: A task force should be organized to propose standards for computer courseware so that they are compatible with the curriculum guidelines.

The second area of concern was the current average quality of classroom teaching. We find that too few teachers are qualified to teach mathematics. We believe that the current conditions for teaching in which too many teachers are underprepared, are isolated in their own classrooms, are overworked and unsupported make it impossible to remedy the quality of teaching without changing the existing staffing pattern. We therefore recommend:

Recommendation 4: In elementary schools, specialist teachers of mathematics should teach all mathematics beginning no later than grade 4 and supervise mathematics instruction at earlier grade levels.

Recommendation 5: In secondary schools, master teachers of mathematics should teach or supervise all mathematics instruction.

Recommendation 6: A task force should be organized to propose certification standards for both elementary school specialist teachers of mathematics and secondary school master teachers of mathematics.

The intent of this group of six recommendations is to assure better prepared teachers working in a more professional environment instructing students with new curricular materials. As conference participants considered how best to realize this intent, related concerns arose about teacher education and about the relevance and use of research-based knowledge in teaching and learning mathematics. The following recommendations were developed to address these concerns:

Recommendation 7: A task force should be organized to prepare model programs for the preservice and inservice education of all mathematics teachers K-12.

Recommendation 8: Research on the learning and teaching of mathematics should be actively encouraged, and research-based knowledge should be incorporated into recommended guidelines and standards.

These eight recommendations require persons or groups carrying them out. Conference participants agreed that substantial improvement requires conviction and action at the local school level, assisted by national coordination and leadership. Two final recommendations reflect this view:

Recommendation 9: In each school or school district, a school mathematics committee should use the curriculum guidelines and staffing recommendations to outline the curriculum and provide support for the mathematics program.

Recommendation 10: Professional organizations concerned with mathematics education should establish a continuing national steering committee for mathematics education to survey efforts of federal, state, and local agencies to reform school mathematics and to report on progress of the reform effort.

These ten recommendations are the steps that must be taken to assure necessary reform of school mathematics. There are barriers to the implementation of these recommendations that are discussed in the body of this report. But, with commitment, hard work, adequate resources, and the leadership of the mathematics education community, all students will have the opportunity they deserve to learn the mathematics we judge necessary for the 1990s.

Introduction

A heated national debate about schooling has ensued from the recent reports critical of current educational practices. The stormy arguments begin with an assumption that "the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people" (National Commission on Excellence in Education, 1983, p. 5). After a decade of neglect, education now has high political and social priority. Reforms are being proposed and debated at all levels. Ideas and recommendations are being discussed by presidential candidates, state governors, legislative bodies, chief state school officers, local superintendents, and school board members.

One area of particular concern discussed in the reports as a target of reform is school mathematics. There is a belief that today, in most classrooms at all school levels, mathematics instruction is neither suitable nor sufficient to adequately equip our children with the mathematical concepts and skills needed for the 21st century. Furthermore, unless something is done to alter current schooling trends, conditions are likely to get worse in the coming decade. Educators will likely respond to the evidence presented in a variety of ways. Some will choose inaction, claiming that change will only make things worse or that this fad will soon pass. Others will react only when the problems are so acute that there is no alternative. And still others will be proactive and see the situation as a challenge and an opportunity to shape the future.

Taking a proactive position, the U.S. Department of Education's Office of Educational Research and Improvement, the National Council of Teachers of Mathematics, and the Wisconsin Center for Education Research jointly sponsored a conference on school mathematics to outline steps and suggest actions which need to be taken. Over 40 invited participants and guests from a variety of professional backgrounds met in December 1983, in Madison, Wisconsin. The object of attention at the conference, school mathematics, was characterized in terms of students' opportunity to learn (curriculum, segmentation, sequencing, etc.), the instructional materials for use in classrooms (texts, courseware, tests, etc.), the intended work of students in performing classroom activities, the education of teachers, and the work of classroom teachers related to students and to the curriculum. The discussion and considerations by the conference participants, the data from background documents, and recommendations from other groups are organized and interpreted under five headings. The first two headings reflect the concerns central to the debate about reform in school mathematics. The questions or concerns raised in each of the five areas are:

Opportunity to Learn Mathematics: Do enough students study enough mathematics? Is the mathematics studied what students need for college and future employment? Are women and minorities discriminated against?

Teaching: Can mathematics teaching be improved significantly? Do most teachers know enough mathematics to teach it well? What do teachers know about calculators and computers and their uses in mathematics? Do conditions and managerial responsibilities associated with teaching make it difficult to teach well?

Teacher Education: Are enough teachers being trained? Could the present model for preparation and inservice education of mathematics teachers be improved? Is it reasonable to expect every elementary school teacher to be well versed in all the mathematics that students might be exposed to? Do we have adequate models or procedures for continuing the education of practicing professionals?

Research: Do current mathematics curriculum programs reflect the knowledge we have gained from research? In what areas is research-based knowledge weak? Where is the greatest payoff for new research?

Coordination: If major changes are appropriate, what assistance is needed? Who needs this assistance? At what level—federal, state, local—should assistance be coordinated? Do we need to be sure that changes actually occur? How might this be accomplished?

The ten recommendations derived from the conference relate to these concerns.

This report presents an analysis of the deliberations of the conference. A companion report documenting the proceedings of the conference has also been prepared. The intent of these two documents is to give direction to educators at all levels in responding to the pressures for change, both by alerting them to needed changes and problems and by proposing strategies, or a sequence of steps, to follow. However, let it be understood that the realization of reform in school mathematics will happen only at the local school level. The steps which are taken by the mathematics education community, in partnership with federal and state agencies, private foundations, schools, publishers, and industries, should be designed to assist schools in making needed reforms.

Opportunity to Learn Mathematics

A principal concern of several recent reports is that many students do not have an opportunity in schools to learn the mathematics they will need for college or employment. Opportunity to learn is determined by four factors:

1. the mathematical topics included in the school curriculum (the content of instruction),
2. the time allotted to each topic in the curriculum,
3. who gets to study which topics (the conditions for course enrollment), and
4. the actions and decisions of teachers during instruction.

The Perceived Problem

To demonstrate the magnitude of this problem, current school practices are described. Then specific concerns are examined and proposals for change identified.

Current School Mathematics: The typical course offerings in mathematics begin with a common curriculum which is followed by most students for seven years. The content of the curriculum is primarily the arithmetic of whole numbers and of positive rational numbers. Other topics such as measurement, integers, and geometry vary from school to school and in general receive too little attention. The emphasis is on getting students to become proficient at a set of procedural pencil-and-paper skills.

This arithmetic curriculum is followed by several options. The traditional college-preparatory two-year curriculum is now taken by about 50 percent of high school students. The content includes a year of algebra, in which the emphasis is on procedures for manipulating algebraic expressions and solving linear equations and inequalities, and a year of (Euclidean) geometry, in which the emphasis is on constructing deductive proofs. For college-bound students who are planning to study mathematics, science, and engineering, a two-year precalculus sequence follows geometry. Calculus, too, is offered at many schools for students who have been accelerated into algebra in grade 8. Finally, for the noncollege-bound students one year of arithmetic (general mathematics, shop mathematics, technical mathematics, etc.) is required, with some of those students then taking college prep courses or other mathematics offerings later in high school.

Overall, by the time an age group of students reaches high school graduation, it is estimated that about 67 percent have taken an algebra course, 50 percent a geometry course, 30 percent an advanced algebra course, 20 percent a senior precalculus course, and 6 percent a calculus course (National Center for Educational Statistics, 1982). Still another view is that about 25 percent of all students take only one year of mathematics in high school (typically general mathematics), about 42 percent take two years (either general math and algebra, or algebra and geometry), and 33 percent take three or more years of math.

Specific Concerns: The general conclusion drawn in the recent reports has been that current course offerings and enrollments in mathematics are inadequate. This is based on four types of information. First, employers and college and university leaders have given testimony about the insufficient mathematical knowledge of recent graduates. For example, between 1975 and 1980, there was a 72 percent increase in enrollments in remedial mathematics courses at colleges. Currently 25 percent of the mathematics courses at public four-year colleges are remedial (National Center for Education Statistics, 1982). Some of the suspected reasons for this situation include an increase of general track students going to college, a change in college expectations (more majors now require mathematics), the mismatch between what is studied in school mathematics and what is needed in some college majors or for many jobs, and possibly some watered down college prep courses to accommodate increasing numbers of unwilling students.

Second, national performance data are bleak. For example, results from the National Assessment of Educational Progress (NAEP) in mathematics (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1981), clearly show that, although most students are reasonably proficient with computational skills, the majority do not understand many basic mathematical concepts and are unable to apply the skills they have learned in even simple problem-solving situations.

Third, when compared with students in other industrialized countries, particularly Japan, American students do not fare well. In large part these results are due to the fact that American students spend less time studying mathematics at all grades and enroll in fewer mathematics courses than do students in other countries. Both Husen (1967) and Wolf (1977), summarizing the international study of mathematics achievement, found student test scores to be highly correlated to the teacher's rating about instructional coverage of the tested topics.

Fourth, the performance and enrollment picture is even more bleak for women and most minorities. For example, on the average, black students complete approximately one year less high school mathematics than their white counterparts (Anick, Carpenter, & Smith, 1981). Both women and most minorities are seriously underrepresented in careers utilizing science and technology. Only 13 percent of the nation's scientists and engineers are women and only 2 percent are black (National Science Foundation, 1982).

A particular concern voiced at the conference was that, while it is clear that schooling practices today are inequitable, this condition is likely to get worse. Affluent suburban school districts already provide their students more opportunities and resources for the study of mathematics, and they are likely to be the first to react to the current crisis and recommendations. They are already spending more money on computers, teacher inservice, and so on, thus widening further the opportunity gap between American schools.

Finally, the participants were aware that current school mathematics programs at all levels simply fail to reflect the impact of the technological revolution now affecting American society. In his remarks at the opening session of this conference, Henry Pollak (1984) provided the following comment: "Two-thirds of all elementary school mathematics is taught in order to make calculators and microprocessors obsolete."

The low cost and availability of calculators, computers, and related new technology has already changed the nature of business, industry, government, and the military. Without considering other concerns, the proposals to reform school mathematics could be based upon the implications of this revolution for classroom instruction. This argument for change has been made from four perspectives.

The first is a utilitarian argument. There is a serious concern about the mathematical needs for employment in the future. Many forecasters are contending that all will need to know more (and different) mathematics for the "high tech" world of tomorrow. The computer and its related uses, such as robotics and lasers, are creating thousands of new jobs. The mathematical requirements will include knowledge of such areas as data handling, exploratory statistics, algorithmic analyses, and mathematical modeling. While we must keep in mind that there are incentives for schooling other than preparation for jobs, preparing all students for this new world is essential. Without such knowledge and skills our students will be unable to participate fully in today's and tomorrow's technological society.

In spite of the attractiveness of this argument, its validity for all students is suspect. For example, the Bureau of Labor Statistics has reported that most 1980 job openings were those with low skill requirements. Furthermore, they project that most openings during the next quarter century will be for low-level service jobs such as janitor, waitress, or cook. In the face of such uncertain predictions about the future job market, our strategy should be to improve the general education for all students so that they will be versatile enough to cope with a rapidly changing employment market.

The second argument is based on the fact that the use and application of mathematics has dramatically expanded. Most areas are not untouched by quantitative techniques. Change has been particularly great in the social sciences and the life sciences. The computer's ability to deal with large sets of information has made quantification possible in such areas as business, economics, linguistics, and sociology. Furthermore, the fundamental mathematical ideas needed in these areas are not necessarily those studied in the traditional course sequence. The algebra, geometry, trigonometry, precalculus, and calculus course sequence was designed with engineering or physical science applications in mind. Today, while this calculus sequence is still very important, its preeminence for all future applications of mathematics is being challenged.

The third argument is based on the fact that the calculator and the computer have changed what concepts and skills should be included in fundamental mathematics courses and what should be emphasized in those courses. Today calculators have replaced human clerks doing calculations in almost every setting. It is time that this should happen in classrooms too. Yet, in arithmetic and algebra (and statistics, if taught) emphasis remains on symbolic manipulations; in trigonometry, the solution of right triangles is still taught. The calculator and computer should be seen as tools to free students from the drudgery of such mathematical calculations and should be used to do most tedious manipulations. The emphasis of instruction should shift to problem solving activities such as formulating expressions and sentences (mathematical modeling) and validating or interpreting results.

Fourth, the computer is changing mathematics by causing new areas to be created (e.g., automata theory) and by making certain topics more important (e.g., combinatorics).

In summary, some students will be producers of mathematical and scientific knowledge; many students will use mathematics in their adult occupations; and all will, or should be, literate consumers of mathematical knowledge and its applications. The aspects of mathematics which are now considered important for all students to learn because of the availability of calculators and computers are not those of the past quarter century.

Proposals for Change: To increase student opportunity to learn mathematics three proposals warrant attention. First, *A Nation at Risk* recommended that all students seeking a high school diploma be required to take three years of high school mathematics (National Commission on Excellence in Education, 1983, p. 24). Most other reports, although not as explicitly, imply that more students should spend more time studying more mathematics.

Second, the Conference Board of the Mathematical Sciences (*The Mathematical Sciences Curriculum K-12: What is Still Fundamental and What is Not*, 1982) has recommended that the content of the existing courses be drastically changed. Three recommendations were made for elementary and middle school mathematics:

- That calculators and computers be introduced into the mathematics classroom at the earliest grade practicable. Calculators and computers should be utilized to enhance the understanding of arithmetic and geometry as well as the learning of problem solving.
- That substantially *more* emphasis be placed on the development of skills in mental arithmetic, estimation, and approximation and that substantially *less* be placed on paper-and-pencil execution of the arithmetic operations.
- That direct experience with the collection and analysis of data be provided for in the curriculum to insure that every student becomes familiar with these important processes (p. iv).

For secondary school mathematics they recommended:

- That the traditional component of the secondary school curriculum be streamlined to make room for important new topics. The content, emphases, and approaches of courses in algebra, geometry, precalculus, and trigonometry need to be re-examined in light of new computer technologies.
- That discrete mathematics, statistics and probability, and computer science now be regarded as "fundamental" and that appropriate topics and techniques from these subjects be introduced into the curriculum. Computer programming should be included at least for college-bound students (p. iv).

The third proposal came from the National Council of Teachers of Mathematics who strongly recommended in *Agenda for Action* (1980) that "Problem solving must be the focus of school mathematics in the 1980s" (p. 2). Furthermore, to meet this recommendation they suggested that "the mathematics curriculum should be organized around problem solving" (p. 2), "appropriate curricular materials to teach problem solving should be developed for all grade levels" (p. 4), and "mathematics programs of the 1980s should involve problem solving by presenting applications at all grade levels" (p. 4).

There was general consensus among the conference participants that change is warranted. The task was to suggest what needed to be done.

It is important to understand that the participants agreed not to debate curriculum goals at the conference because they were in general agreement on them. Three previous reports—*An Agenda for Action*, prepared by the National Council of Teachers of Mathematics (1980); *The Mathematical Sciences Curriculum K-12: What is Still Fundamental and What is Not*, prepared by the Conference Board of the Mathematical Sciences (1982); and *Academic Preparation for College*, prepared by the College Entrance Examination Board (1983)—provide the base upon which content outlines can be built. Then, in addition to these three documents, goal statements are provided in the recommendations from various commission reports, reports from other recent conferences in part leading to the same concerns, and a plethora of recent thought-provoking papers on aspects of goals and content. Although continued debate on goals is important, the mathematics education community is currently in agreement about goals, and we should get on with the task of trying to reach them. Setting guidelines for reaching goals and stating expectations about goals is our next task.

Needed Actions

Several needed actions based on these proposals are apparent. For example, state legislators and officials in state departments of education must establish new graduation and course requirements. Furthermore, with local school districts, they must find ways in which needed resources can be found for implementing these changes. And school boards and administrators must plan steps to accomplish these changes.

The conference participants felt that the mathematics education community had both the responsibility and the expertise to suggest a strategy for the preparation of new curricular materials.

The strategy being suggested for the preparation of new materials is, in our judgment, the most viable of three options. First, it is *not* being suggested that a new curriculum development effort be started similar to that followed in the post-Sputnik era. While good materials were rapidly prepared by the School Mathematics Study Group (SMSG), they were not used as intended in most classes. The "center to periphery" notion of development and implementation rarely works as intended, because teachers too often have no sense of ownership of the materials. As one teacher put it, "I teach your mathematics to my children" (Stephens, 1982).

Nor are we suggesting that materials be locally developed. In the past, such efforts have usually resulted in a "cut-and-paste" program which is inadequate and incomplete. Preparing good materials which are conceptually sound and comprehensive is an expensive, time-consuming task.

The strategy we are recommending is for commercial publishers to prepare materials following a set of guidelines which now need to be developed. The difficulty of this strategy is that it involves two parties—publishers and school staffs. Publishers will produce texts, tests, manipulatives, software, and so on, if they are certain that there is a market. But, in the past when new materials were developed they were difficult to sell. In any market the primary responsibility for change rests with the consumer. Thus, the guidelines to be developed must be used by school teachers and administrators to assist them in planning the mathematics curriculum of the future and as a basis for demanding new materials from publishers. Furthermore, they should eventually refuse to purchase and use materials that fail to meet those guidelines.

What is needed is neither a list of basic skills such as those advocated in the 1940s and later, nor the "modern math strands" of the 1960s, but a set of curriculum guidelines for school mathematics. These curriculum guidelines should include a detailed outline of the suggested mathematical topics to be taught at each grade, an estimate of the time to be spent on each topic, options for including topics, options for sequencing topics, the way topics should be organized for instruction, possible materials for instruction, the instructional methods which teachers might use for those materials, and suggested standards for student performance following instruction. Furthermore, the guidelines should suggest the features for concrete manipulatives, calculators, tests, and software as well as texts.

The task of preparing such guidelines can be separated according to the elementary curriculum, the secondary curriculum, and courseware.

Recommendation 1: A task force should be organized to propose guidelines for a K-8 mathematics curriculum.

Recommendation 2: A task force should be organized to propose guidelines for a 7-14 mathematics curriculum.

Recommendation 3: A task force should be organized to propose standards for computer courseware so that they are compatible with the curriculum guidelines.

The rationale for these recommendations is that a group of persons with a wide variety of backgrounds and interests in school mathematics can best prepare guidelines. The document should be a practical resource guide for developing a curriculum for the 1990s. The group should include teachers, mathematicians, and educators, with persons from industry, administrators, publishers, and psychologists serving as consultants at various stages of development. The rationale for three different task forces is both to separate this large job into three smaller, more manageable jobs and to acknowledge the different problems in the three tasks. The new elementary school curriculum should involve both the specification of a somewhat common program for all students and an increased use of calculators. For the new secondary program, a flexible and integrated approach to mathematical topics is needed so that all students have an opportunity to study some mathematics at least through grade 11. This will involve a restructuring of the traditional algebra-geometry-advanced algebra-precalculus-calculus sequence of courses.

Finally, a separate task force on computer courseware is recommended. Recently, such material for use in school mathematics has been proliferating rapidly. However, most current courseware lacks both content and pedagogic validity. The need for courseware standards and compatibility of courseware with each curriculum guideline seems obvious.

Each task force should meet for at least six weeks and possibly for as long as eight weeks during each of two years. The interim period would allow for an initial dissemination of the guidelines and permit feedback from a variety of groups. The second meeting would allow for revising and polishing each set of guidelines for final dissemination.

Prior to the first meeting of each task force, information should be gathered about both current practices and alternatives in this and other countries with respect to the scope and sequence of mathematical topics in the curricula. Also, recommendations from scholarly groups, industry, and interested parties regarding their mathematical expectations for students in grades K-14 should be collected.

The expectation is that the task forces will begin with a considerable database derived from the deliberations and recommendations of others.

The general concern that needs to be addressed by the *elementary school guidelines* is the current dependence on paper-and-pencil arithmetic, much of which is today obsolete. This dependence is a major barrier to the implementation of other recommendations that have been made for school mathematics. That is, a decreased dependence upon paper-and-pencil arithmetic would seem to be a prerequisite for updating and improving the elementary school mathematics curriculum. At the opening session of the conference, Willoughby (1984) said, "We must encourage teachers to teach general principles . . . . teaching children to think, to continue to learn, and to be able to deal with new situations and new conditions."

The dominance of the curriculum by paper-and-pencil skills applies more than just to what is taught. Topics are sequenced in light of paper-and-pencil computation and are often approached pedagogically in ways affected by computation. Thus, repetition of abstract routines has overshadowed meaningful learning. Freeing the curriculum from paper-and-pencil dominance could have profound implications for the timing and approaches given to topics that remain. For example, rather than being driven by algorithms, the sequence should be determined by the readiness for applications, and children could have the concrete experiences necessary for meaningful abstractions. The placement of topics will need to be reconsidered.

One method to correct these problems is to use calculators at all levels. Then new topics, now considered fundamental, could be provided. Such topics include problem solving, measurement, applications, geometry, estimation, and working with data.

The *secondary school guidelines* should address the following:

1. The topics should be unified and integrated so that the inter-relationships of algebra, geometry, and applications are made.
2. Calculators and computers should be available for all students and teachers, and they should be used when appropriate.

Finally, the recommendation of increasing the amount of mathematics for all students should not result in keeping all students in the same traditional precalculus courses. This intent cannot be met by adding more "general mathematics" courses or shifting lower track students into a two-year beginning algebra course. Alternate programs need to be defined and materials identified or developed.

The *courseware guidelines* should address the following questions:

1. Which mathematics topics, identified in the first two guidelines, can be well supported with calculators or microcomputers, and how can appropriate software be developed?
2. What types of software are appropriate for different modes of instruction (teacher demonstration, whole class interaction, small groups, individual)?
3. How does one evaluate different possible uses of existing software and relate them to the curriculum?
4. How can we as teachers communicate what is available now and what is possible? How can we become open to learning and renewing as future generations of software, languages, expert systems, and hardware are developed?
5. How can the problem of differential access to computers be overcome?

Although adequate access is tangential to this task force's goal of developing standards for courseware, we believe that all students should have access to computers to enhance their mathematical learning and to do their mathematical work. Having access to a computer is not of itself insurance of an education. There must be appropriately prepared staff and adequate materials to integrate the technological advances with the school mathematics program.

The proposed strategy is to have this task force begin with the two prior curriculum guidelines. Although the appropriate use of calculators and computers is expected to be an integral part of the deliberations of the previous two task forces, because of the nature of the above questions a separate task force to consider calculator/computer courseware is suggested.

In summary, the conference participants felt that a much larger portion of the population will have the opportunity to learn more (and somewhat different) mathematics in the 1990s than ever before. To reach this objective several actions need to be taken. One important action is the preparation of curriculum guidelines and standards for computer courseware. Such guidelines are necessary if actual reform is to occur.

Teaching

A second area of concern addressed by the recent reports and the conference participants is the quality of teaching, including the professional status of teachers and the low level of mathematics preparation and training of many teachers. It was assumed that better mathematics teaching is possible, but the need to improve is compounded by the need for curricular change. Here distinctions between the planned curriculum and the actual curriculum, and between upgrading content and upgrading the instructional environment, should be made. Changing the content of the curriculum, as addressed in the last section, is necessary. But that is not sufficient to ensure that students will be more knowledgeable. Proposed changes in content and pedagogy only can be implemented by teachers and administrators within a truly professional milieu, which at present simply does not exist.

The Concerns

To understand the concerns about teaching and the need for improvement, pictures of the job of teaching, teachers' professional status, and expectations are given.

The Job of Teaching: In 1975, the National Advisory Committee on Mathematical Education (NACOME) commissioned a study of elementary school mathematics instruction. The picture drawn from this survey is as follows:

> The "median" classroom is self-contained. The mathematics period is about 43 minutes long, and about half of this time is written work. A single text is used in whole-class instruction. The text is followed fairly closely . . . which for students is primarily a source of problem sets. (Conference Board of Mathematical Sciences, 1975, p. 77)

And the following remarks from one of the NSF case studies (Stake and Easley, 1978) of mathematics teaching describe secondary education:

> In all math classes I visited, the sequence of activities was the same. First, answers were given for the previous day's assignment. The more difficult problems were worked by the teacher or a student at the chalkboard. A brief explanation, sometimes none at all, was given of the new material, and problems were assigned for the next day. The remainder of the class was devoted to working on the homework while the teacher moved about the room answering questions. The most noticeable thing about math classes was the repetition of this routine. (Welch, 1978, p. 6)

Romberg and Carpenter (in press) identified three serious limitations of mathematics instruction as characterized above.

First, mathematics is assumed to be a static bounded discipline. The emphasis is on teaching fragmented concepts and skills associated with aspects of this discipline. This fragmentation of mathematics has divorced the subject from reality and from inquiry. Essential characteristics of mathematics such as abstracting, inventing, proving, and applying are often lost.

Second, the acquisition of concepts and skills becomes an end in itself, and students spend their time absorbing what other people have done, rather than in having experiences of their own. Yet, current research indicates that acquired knowledge is not simply a collection of concepts and procedural skills filed in long-term memory. Rather the knowledge is structured by individuals in meaningful ways, which grow and change over time.

Third, the role of teachers in the traditional classroom is managerial or procedural in that "their job is to assign lessons to their classes of students, start and stop the lessons according to some schedule, explain the rules and procedures of each lesson, judge the actions of the students during the lesson, and maintain order and control throughout" (Romberg, in press, p. 13).

In such situations, the teaching of mathematics is too often done without care or reflection. The job of teaching is perceived to be procedural or managerial and not adaptive. Too many teachers feel obligated to cover the book. They may adapt instruction so they can better manage the diverse group of students in their class or so their students will have higher test scores. Too few teachers see that student learning of mathematical methods and their use in solving problems is the primary goal of instruction. At the elementary school level most teachers have an inadequate mathematical background. About one-half of the states require no mathematics content or methods course for elementary school certification. It is unreasonable to expect that teachers who have no knowledge or preparation in mathematics or the teaching and learning of mathematics will perform adequately in classrooms. Now growing numbers of teachers at the secondary school level also are underprepared. To meet current shortages, many teachers are now being licensed with minimal preparation. This problem can only get worse during the next decade if the current trends in teacher education continue.

Furthermore, teachers tend to be isolated in their own classrooms. They have little opportunity to share information with other staff members and little access to new knowledge (Tye & Tye, 1984). Unless the proposed reforms become recognized, understood, and owned by teachers and teachers understand and are valued for their role in the reform movement, real change is unlikely to occur.

Professional Status: Most teachers want to act as and be treated as professionals, even if the working conditions in many schools make it nearly impossible. There are too few adults trying to control a large number of students. The job structure in schools provides for little differentiation. Thus, each elementary teacher is expected to do all—teach every subject equally well, lecture, plan, supervise a lunch hall, and so on. Similarly, secondary teachers teach five or six classes, often with three or four preparations, as well as carry out other duties. There is usually not enough time to reflect or plan. Hard and productive work is often not rewarded; nor is time to retrain or refresh generally supported. Furthermore, many teachers are leaving the classroom for higher-paying jobs, and fewer students are now entering teaching. During the 1970s, there was a 77 percent decline in the number of high school mathematics teachers being prepared. In 1981, 43 states reported a shortage of mathematics teachers, and by some estimates 50 percent of beginning mathematics teachers are not fully certified to teach mathematics (Hurd, 1982). Thus, we need to change not only what mathematics is being taught but also the professional status and qualifications of mathematics teachers.

Expectations: It is anticipated that the mathematics teachers in the 1990s not only will be teaching a new mathematics program, they also will be using a different pedagogy and creating a different instructional environment. Such instruction will be carried out with understanding, skill, and professional enthusiasm.

Using the new curriculum materials will involve much more than replacing one set of materials with another similar set. The use of concrete materials, calculators, computers, and group activities involving real data gathering will replace paper-and-pencil problem tasks. If current research-based knowledge on teaching is followed, teachers will spend more time planning lessons (Shavelson, 1976), developing the ideas in each lesson (Good, Grouws, & Ebmeier, 1983), and directing small group problem solving (Webb, 1982). Furthermore, emphasis will be on the strategies used to solve the problems rather than just on right answers (Case, 1978).

Needed Actions

Several needed actions based on this evidence about conditions of teaching and expectations are apparent. Communities and school administrators need to create a more professional environment for teaching. Time to plan, interact with others, refresh, teach with other teachers, and so on is essential. State and local leaders need to find resources so that such conditions are possible.

With respect to mathematics instruction, it is hoped that all mathematics would be taught or supervised by well qualified teachers. There are at least two options which could be followed to meet this expectation. The first would be to organize a massive effort for the preservice and inservice education of all teachers leading to a reasonable level of mathematical competence and providing each a communication structure for sharing ideas. This option, however desirable, is too costly. The second and more pragmatic strategy would be for schools to adopt a differential staffing pattern so that mathematics instruction is carried out or directed by adequately prepared capable teachers. Thus, the following three recommendations are made:

Recommendation 4: In elementary schools, specialist teachers of mathematics should teach all mathematics beginning no later than grade 4 and supervise mathematics instruction at earlier grade levels.

Recommendation 5: In secondary schools, master teachers of mathematics should teach or supervise all mathematics instruction.

Recommendation 6: A task force should be organized to propose certification standards for both elementary school specialist teachers of mathematics and secondary school master teachers of mathematics.

The general rationale for the creation of special teachers of mathematics is based on the belief that improved curriculum and instruction in mathematics at any school level requires staff who are prepared in mathematics as well as in teaching and in student learning. At present there is a mismatch between preparation and expectations for the teaching of mathematics at any school level. The present demands on a typical self-contained classroom teacher at the fifth-grade level are too diverse and intensive to allow time for also providing adequate instruction in mathematics. The level of sophistication of the content and subject matter taught requires individuals who have the capabilities of a special mathematics teacher. The designation of special mathematics teachers for mathematics at the elementary school level and master teachers at the secondary school level is a realistic approach to solving this problem.

In addition, the creation of specialist positions provides an attractive professional opportunity. Practicing teachers who have background and interest in the teaching and learning of mathematics will have new opportunities for responsibility in an area of expertise and interest. These positions will motivate new or prospective teachers to select specialized or additional training.

Program improvement in mathematics requires monitoring, accountability, and a mechanism for continued improvement. A dynamic rather than a static system of improvement is required. The special mathematics teachers should be responsible and accountable for this monitoring and continued improvement of curriculum and instruction in mathematics. Differential staffing provides a strategy for schools and school districts to insure that they are teaching appropriate mathematical skills in effective and affective ways so that children can learn in today's schools the skills that will be useful in tomorrow's society.

To realize the full benefit of having special mathematics teachers there must be commitment on the part of the school administration to the teachers involved. The administration must be willing to support the special mathematics teachers by providing time and resources for these teachers to develop, coordinate, and monitor the mathematics programs in their buildings. (See recommendation 9 for additional discussion of coordination at the school or district level.)

While each special mathematics teacher is expected to provide direct instruction for some students, each must also have sufficient time allocated for the other aspects of the job. It is reasonable to expect that 40 to 50 percent of their time would be spent on staff development, demonstration teaching, diagnosis and remediation, curriculum planning, personal professional growth, and monitoring the overall mathematics program.

Finally, the opportunity and financial support for special mathematics teachers to belong to and participate in professional organizations must be provided. Such involvement is essential to the continued professional growth and development of the special mathematics teachers themselves. The promise this holds is that mathematics curriculum and instruction will, through these special teachers, become a system that is more responsive to new ideas and developments.

The strategy for assisting schools in meeting both recommendations is to have a task force propose standards for specialists' and master teachers' preparation, licensing, job responsibilities, and so on. The work of this task force would be similar to that of past committees of the National Council of Teachers of Mathematics and the Mathematical Association of America. Participation of state education agencies and organizations that develop teacher certification standards is essential.

Reaching these goals will take time. However, a timeline and a sequence of steps which could be followed need to be specified so that in each American school mathematics instruction will be carried out by well-trained teachers in the near future.

In summary, combining new curriculum materials with better prepared teachers operating in a more professional teaching atmosphere should be the overall goal of this reform effort.

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The Impact of Computing Technology on School Mathematics

Report of an NCTM Conference
National Council of Teachers of Mathematics

One of the critical problems facing mathematics education today is the need for curricula and instructional methods to respond to the influence that computing technology is having on mathematics and its teaching. Numerous conferences, reports, and projects have focused on different aspects of the effect of emerging technology on mathematics curricula and instruction. For example, the Conference Board of the Mathematical Sciences produced a set of guidelines that identified fundamental topics in K-12 mathematics (CBMS 1983), and an NSF-sponsored conference at the University of Maryland resulted in a document that provided insight into the effect of the computer on subjects in the traditional secondary mathematics curriculum (Fey 1984).

NCTM has made many efforts to provide leadership as decisions about the impact of computing technology on mathematics education are made. In 1981 its Board of Directors established the Technology Advisory Committee, which has subsequently engaged in several projects and activities that serve members in the area of technology and which advises the Board on matters relating to technology and mathematics education. NCTM sponsors a continuing series of seminars for K-12 mathematics teachers on the uses of computers in mathematics instruction and has for a long time devoted program slots at regional and national meetings to issues associated with the impact of technology on mathematics education. The publications of NCTM, including the journals and the 1984 Yearbook, reflect the continuing concern of the organization with this use.

In an effort to continue its leadership role, NCTM and the Center for Mathematics Education of the University of Maryland, with financial support provided by the National Science Foundation, sponsored a conference at NCTM headquarters during March 1984 to consider "The Impact of Computing on School Mathematics." Participants represented a number of interested constituencies: government funding agencies, commercial publishers of mathematics materials, the standardized testing industry, mathematics teacher educators, mathematicians, computer scientists, elementary, middle, and secondary mathematics teachers, and district- and state-level mathematics supervisors. Participants were asked to respond to a series of questions related to the impact of computing technology on the mathematics curriculum, mathematics instruction, and mathematics teacher education, and to formulate general recommendations in each of these areas. The conference produced the recommendations herein—each gaining broad, if not unanimous, support.

Recommendations on Curriculum, Instruction, and Teacher Education

Prologue

Applications of calculators, computers, and other electronic information technology are reshaping the fundamental methods of doing and teaching mathematics. When used as tools for arithmetic and for the analysis of graphic or symbolic data, calculators and computers offer powerful new approaches to familiar problems and access to entirely new branches of mathematics. Applications of these same capabilities to instruction are bringing major changes to mathematics classrooms and the roles of mathematics teachers.

The major influence of technology on mathematics education is its potential to shift the focus of instruction from an emphasis on manipulative skills to an emphasis on developing concepts, relationships, structures, and problem-solving skills. Traditional precollege mathematics curricula have stressed the development of a variety of mechanical procedures, including the computational algorithms of arithmetic and the transformation of symbolic expressions in algebra, trigonometry, and analysis. The use of calculators and computers as standard tools in quantitative problem-solving situations, however, has diminished the value of human proficiency in the execution of such procedures. Much of the instructional time currently devoted to acquiring proficiency with paper-and-pencil algorithms should be reallocated to support a range of new or previously neglected topics that have a valid place in the K-12 mathematics curriculum. Moreover, teacher education programs must be modified to reflect these changes in school mathematics content and to model the delivery of instruction through appropriate applications of technology.

The proposals that follow are intended as guidelines for selecting the content of precollege mathematics curricula, for teaching that content in a manner that takes advantage of emerging technology, and for designing teacher education programs that recognize the changing curricular patterns and instructional roles for teachers. The proposals are based on five fundamental assumptions:

1. Coordinated change can take place simultaneously at all levels of mathematics instruction.
2. All students and teachers will have access to calculators and computers for the study of mathematics, in the classroom and at home.
3. All students will experience appropriate application of computers in the study of each school discipline.
4. State/provincial, district, and local mathematics curriculum guidelines and criteria for mathematics textbook adoption will be rewritten to reflect the changing priorities of school mathematics.
5. The publishers of standardized tests and instructional materials will
    (a) immediately begin developing products that are consistent with the changing objectives of precollege mathematics;
    (b) continue support for that development on a schedule that facilitates the implementation of proposed curricular changes.

Technology and the Mathematics Curriculum

Today, the computational skills of arithmetic, algebra, geometry, trigonometry, and calculus dominate the K-12 mathematics curriculum. The content and sequences of courses are planned carefully so that students acquire intricate hierarchies of prerequisite skills for each major computational algorithm. Although these well-known mathematical procedures originated as essential aids to efficient problem solving, most of the algorithms of school mathematics have now been programmed for rapid execution by calculators and computers.

To do arithmetic today, mental operations are best for obtaining quick approximations; calculators are the tools of choice for one-time computations; and computers are most appropriate for repetitive calculations. In algebra, trigonometry, and calculus, computers can execute the numerical and symbolic manipulative procedures that students spend countless hours mastering.

As a consequence of this changing environment for mathematical work, the curriculum in grades K-12 needs careful reassessment and revision. Classroom teachers, mathematics supervisors, members of state departments/provincial ministries of education, authors and publishers of textbooks, developers of standardized tests, and other curriculum developers must consider the following recommendations and the questions they raise.

Elementary School Level

The elementary school mathematics curriculum has traditionally focused on developing students' skills in computing with whole numbers, fractions, decimals, and percents. Since computers and calculators can perform such operations more quickly and accurately than can usually be done otherwise, the traditional goals of elementary school mathematics must be reexamined and the predominance of computation-related objectives must be reassessed. Curriculum developers are urged to consider the following recommendations that suggest a broadened view of mathematics appropriate for grades K-4.

- Calculators should routinely be available to students in all activities associated with mathematics learning, including testing. Students should be taught to distinguish situations in which calculators are appropriate aids to computation from situations in which mental operations or paper-and-pencil computations are more appropriate.

- Emphasis should continue to be placed on students' knowledge of basic facts required for proficient mental arithmetic and estimation. However, significant portions of elementary curricula devoted to algorithms for multiple-digit calculations can be eliminated.
- Instruction must shift to emphasize the meaning of arithmetic operations. Such understanding is essential for problem solving.
- Experience with physical manipulatives and other concrete representations of concepts must continue to be an important phase of learning mathematical ideas. Because of the increased instructional emphasis on meaning and understanding, this activity must not be overlooked as the curriculum evolves to take advantage of computers and calculators for instruction.
- Because computers and calculators can be used effectively in teaching mathematical concepts, no prior assumptions should be made about the appropriateness of any given mathematical topic for elementary students. For instance, decimals, negative numbers, and scientific notation appear naturally when using calculators and can be taught as they arise. Computers facilitate an early introduction to geometric concepts such as transformations, congruence, and vectors; statistical concepts such as randomness; and algebraic concepts such as variable and function.
- Preprogramming activities and simple computer programming in Logo or BASIC can be done by students as early as the kindergarten level to convey both mathematical and computer concepts. Computer literacy should come as a natural by-product of such experiences rather than as a special addition to the elementary school curriculum.

Middle School Level

The mathematics curriculum for the middle school (grades 5-8) must take into account incoming students' knowledge of new topics such as computing, their understanding of topics formerly reserved for the middle grades, and their more limited skills in topics such as arithmetic algorithms. In addition to building on the changes proposed for the elementary grades, curriculum developers for the middle grades should consider the following specific recommendations.

- Mathematics in the middle grades should emphasize the development of "number sense"—the intuitive feeling for the relative sizes of numbers that is essential in skillful estimation, approximation, mental arithmetic, and the interpretation of results for reasonableness.
- Some portion of instructional time should be given to the study of discrete mathematics, including counting, graph theory, probability, and logic, which is important and appropriate for the middle grades.
- Calculators and computers can be used to teach iterative procedures for solving significant problems before traditional formal methods are presented. Such experiences should be part of the middle school mathematics curriculum.
- An introduction to statistics should include extensive gathering, organization, and presentation of data. Important concepts can be developed in the context of real data sets whose manipulation and examination is aided by computer analysis and graphing software.
- Middle school mathematics programs should take advantage of the visual display capabilities of computer graphics that support and underscore the importance of informal geometry objectives. Transformations, mensuration formulas, and spatial visualization can be vividly illustrated by using computer graphics.
- Increased emphasis should be placed on such nontraditional methods of problem solving as organized lists, guess and check, geometrical sketches, and successive approximations, all of which are made feasible by calculators and computers.
- Computer programming experiences that introduce the concepts of variable and function should be provided. These experiences should help prepare students for the study of algebra.
- By the end of grade 6, students should be able to write simple computer programs that require looping and branching concepts. The emphasis in such programming activities should be on problems that convey significant mathematical ideas.

Senior High School Level

Developers of senior high school mathematics curricula must consider both the changing preparation of their entering students and the changing college and work environments their graduates will enter. The following recommendations are suggested as guidelines for the curricular evolution that must occur if high school mathematics programs are to reflect appropriately the influence of technology on mathematics and its teaching.

- The selection and sequencing of high school mathematics for college-bound students should no longer be governed solely by preparation for calculus. The most important mathematics for many students will consist of topics from discrete mathematics and statistics. High school programs must reflect these goals in their content and priorities.
- Computing is changing calculus and its traditional prerequisite subjects. Computer symbolic systems, graphics, and numerical analysis software make student mastery of manipulative procedures less important and, at the same time, offer dynamic tools for the illustration of fundamental concepts and processes. Such changes should be considered as the influence of calculus on high school mathematics is assessed.
- The content priorities and approaches to topics in high school geometry should be reconsidered in light of computer graphics and their applications. The geometry of three-dimensional space, trigonometry, vectors, coordinates, and transformations are all made more accessible with computer assistance through visual displays and complex calculations.
- The skill objectives of algebra must be reassessed to identify those procedures more easily done by computer (e.g., combination of expressions or solution of equations) or calculator (e.g., values of trigonometric functions, logarithms, or exponentials). The properties of elementary functions are still important for modeling quantitative relationships, but proficiency in many familiar computational processes is of little value.
- In many topics of high school mathematics, computers and calculators can be used to discover and test principles and methods. For example, concepts and theorems can be illustrated numerically and graphically to develop sound understanding before formal proof is attempted.
- For high school students with limited ability or interest in mathematics, technology offers an opportunity to enrich curricula with realistic problem-solving situations without insisting on a mastery of skill prerequisites. For example, these students can concentrate on the skills needed to collect, organize, and interpret mathematical information, skills appropriate for estimation and approximation, problem-solving strategies, data analysis methods, and reasoned evaluation of results.

The time needed for new curricula will probably exceed the time that can be saved by reduced emphasis on topics of diminished importance or by a more efficient organization of instruction. Mathematics is of such fundamental importance, however, that schools must allocate more time to its study. This means more minutes per day in the elementary and middle grades and more years of mathematics in secondary school.

Technology and Mathematics Instruction

The impressive capabilities of emerging technologies promise major changes in the organization of mathematics instruction and in the roles of students and teachers. Computers offer a dynamic electronic chalkboard for the demonstration of mathematical ideas. As laboratory tools, they assist in the exploration and discovery of concepts, the practice of skills, applied problem solving, and instruction by simulation. Computer-based evaluation and information management provide further opportunities for change in traditional teaching/learning patterns.

Although there may be risks associated with changing instructional patterns and teachers' and students' roles, the potential for improved instruction in mathematics makes the search for ways to reduce those risks worthwhile. If teachers and students are to exploit the potential of technology to support instruction, they must become adept at the new roles described in the recommendations below.

One of the most immediate consequences of the integration of technology into mathematics instruction is the increased need for teachers to make informed decisions about questions that relate to certain aspects of the mathematics curriculum. The following recommendations focus on these issues.

- Teachers must be able to make informed curricular decisions about the appropriate place and strategy for using technology to enhance instruction.

- Teachers must learn how dynamic visual displays made possible by computers can aid transition from concrete experience to abstract mathematical ideas.
- Teachers must become adept at interpreting and elaborating computer-based evaluation of student learning and in using information management systems that organize curriculum options and assessment data.

New roles for classroom teachers and students are also made possible by technological enhancements of mathematics instruction. Among these roles are the following:

- Teachers must become effective catalysts for student-directed learning. They must be able to pose stimulating problems and to probe student understanding with questions that begin "What would happen if . . .?"
- Teachers must serve as learning models in attacking mathematical questions. They must be comfortable assisting students in learning situations in which the topic being studied or the resource employed is unfamiliar. Teachers must be prepared to say, "I don't know yet," while they join students in exploration.
- Teachers must become facilitators of cooperative learning as students engage in realistic learning projects that computers make attractive. Teachers and students must emulate real problem solvers as they deal with problems in complexity, accuracy, and precision that occur in the collection and analysis of real data.
- For students, the greatest promise and challenge of technology in mathematics learning is the move toward more self-directed and self-monitored learning. Students must become adept at using computer-based instructional materials, mathematical tool software, and personal programming skills to develop and practice the use of mathematical concepts, principles, and problem-solving processes.
- Student-directed learning suggests that students may become dispersed throughout the learning environment. Teachers must know how to maintain effective communication with and among learners and must recognize that individualized technology-enhanced learning is not synonymous with independent study.

The fundamental theme that underlies recommended changes in patterns of mathematics instruction and in the roles of teachers and students is the emergence of a new relationship among teachers, students, and the subject matter. The presence of computers and calculators in demonstration, practice, problem solving, and evaluation creates a new classroom dynamic in which teachers and students are natural partners in the search for an understanding of mathematical ideas and a solution of mathematical problems.

Technology and Mathematics Teacher Education

The recommendations in previous sections of this report suggest a new look to mathematics classrooms, with concepts and problem solving the focus of curricula and teachers joining their students as active partners in learning. If these changes are to occur, classroom teachers must be the primary agents of change. Consequently, preservice and in-service mathematics teacher education programs must be restructured to prepare teachers for their new roles.

Computer education for teachers of mathematics requires understanding and skill in three areas: (1) the uses of computers and calculators as problem-solving tools in mathematics; (2) the uses of computers for the presentation, evaluation, and management of instruction; and (3) the concepts of computer literacy that contribute to, or depend on, the knowledge of mathematics. The breadth and depth of appropriate computer education will vary with the teacher's grade level and subject specialization. The following recommendations provide guidelines to the content and form of preservice and in-service experiences suitable for elementary, middle, and senior high school mathematics teachers.

Mathematics content and methods courses in teacher education programs must be examined and restructured to give more attention to priority content areas and to make calculator and computer use an integral part of all such courses. In particular, the following recommendations apply:

- Students preparing to teach at the elementary level must complete the mathematics requirements outlined in the *Guidelines for the Preparation of Teachers of Mathematics* (NCTM 1981), with special emphasis on estimation, approximation, mental arithmetic, data organization and analysis, informal geometry (using a variety of approaches to the subject that reflect its visual nature such as transformations, vectors, and coordinates), probability concepts, and measurement concepts.

- Computer literacy and programming concepts needed by elementary teachers should be acquired through experience in the mathematics content and methods courses. Consequently, teachers preparing to teach at this level need not be required to take a formal computer science course, although a course focusing on computer applications in education is recommended.
- Students preparing to teach at the middle school level should also complete the mathematical requirements outlined in the NCTM *Guidelines* for this level, with special attention to the priority topics recommended for elementary teachers and experience with more advanced topics from algebra, analysis, statistics, and discrete mathematics.
- Middle school mathematics teachers should complete a course on computer applications in education and a computer science course that focuses on problem solving and, more specifically, on the nature of algorithmic thinking and the computer implementation of problem solutions.
- Students preparing to teach mathematics at the secondary level should complete the mathematics major outlined in the NCTM *Guidelines* and coursework in discrete mathematics, probability and statistics (including exploratory data analysis), and numerical analysis. All topics must be taught in ways that demonstrate appropriate applications of computing to mathematics.
- In addition to the computing courses recommended for middle school teachers, secondary mathematics teachers should complete study that provides further experience with structured programming, algorithmic thinking, mathematical modeling, and other mathematics fundamental to computer science.

To provide teachers of mathematics with the knowledge and skills needed to use computers and other technology effectively for instruction, programs of preservice and in-service education must include the following components:

- Level 1 (awareness). Through the use of computers, calculators, and other technology as an integral part of the delivery of instruction in mathematics, teachers must be exposed to a variety of applications that enhance their understanding of the potential of these technologies. They must experience a range of hands-on activities and explorations and see demonstrations of appropriate uses of technology in classrooms.
- Level 2 (interaction with prepared software). Teachers at this level must be given training in the selection and use of appropriate software to enhance instruction in mathematics, and information should be given to them about sources of mathematics-related software. They must develop, deliver, and evaluate lessons and units that integrate existing software into mathematics instruction.
- Level 3 (interaction with programming). Teachers should be given the opportunity to engage in programming-related activities such as tracing, debugging, or modifying existing programs, all of which can enhance mathematics learning. Such experiences should focus on how programming ideas can be used to teach mathematical concepts, not on developing the teacher's personal programming skills.

The most important point in these recommendations for change in mathematics teacher education is the requirement that preservice and in-service programs must model the desired changes. Teachers must *experience* mathematics content and methods courses that emphasize priority topics and that present those topics using a variety of techniques that fully integrate computers, calculators, and other information-processing technologies. The realization of this goal will require commitment and cooperation among mathematics and teacher education faculties. Institutions must commit needed resources for equipment and must support faculty study time.

- Every institution should identify a computer resource teacher who can serve as a technical advisor for all teachers and a mathematics/computer resource teacher who can provide leadership in helping teachers make appropriate use of computing devices in mathematics instruction. Institutional support for training such individuals will be needed.
- Institutions must provide ongoing support mechanisms. Administrators must be educated about proposed changes in mathematics content and the

delivery of instruction and must be supportive of changes. Initiative and achievement must be recognized.

- Colleges and universities should work with local schools and districts to devise creative models for inservice programs to upgrade teachers' backgrounds and ways to provide support for teachers as changes are implemented. Such programs must build in mechanisms to inform teachers at upper levels about curricular and instructional changes at lower levels and vice versa.
- Professional organizations such as NCTM must offer training workshops to prepare teacher educators for leadership roles in effecting change. Funding agencies must offer support for retraining activities.

Well-educated, committed teachers will be the key to the successful implementation of curricular and instructional improvements. The computer education of preservice and inservice teachers represents a critical investment necessary for change to occur.

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Academic Preparation for College*

The College Board

Mathematics

Why?

All people need some knowledge of mathematics to function well in today's society. Mathematics is an indispensable language of science and technology as well as business and finance. All people, therefore, need some fluency in this language if they are to contribute to and fare well in our contemporary world.

More than at any time in the past the knowledge and appreciation of mathematics is essential to students' intellectual development. The advances of recent years in computer science and other highly technical fields such as space science have opened new horizons to those trained in mathematics. Young people who avail themselves of the opportunity to gain strong preparation in mathematics and in the sciences not only will grow intellectually but also keep open the door to a wide range of career choices.

Students going to college need mathematical skills beyond the elementary ones. They need a knowledge of computing to deal with the new age of computers and information systems. They need a knowledge of algebra, geometry, and functions to major in a wide range of fields, from archaeology to zoology. They need a knowledge of statistics for such fields as business, psychology, and economics.

More extensive knowledge and skills, including preparation for calculus, will be needed by college entrants who expect to take advanced mathematics courses or to major in such fields as engineering, economics, premedicine, computer science, or the natural sciences.

What?

The following learning outcomes in some cases provide greater detail concerning the Basic Academic Competency in mathematics outlined in Chapter II and the computer competency described in Chapter III, but in other cases go beyond the competencies to specify further outcomes of the study of mathematics. College entrants will need the following basic mathematical proficiency.

- The ability to apply mathematical techniques in the solution of real-life problems and to recognize when to apply those techniques.
- Familiarity with the language, notation, and deductive nature of mathematics and the ability to express quantitative ideas with precision.
- The ability to use computers and calculators.
- Familiarity with the basic concepts of statistics and statistical reasoning.
- Knowledge in considerable depth and detail of algebra, geometry, and functions.

More specifically, college entrants will need the following preparation in mathematics.

Computing

- Familiarity with computer programming and the use of prepared computer programs in mathematics.
- The ability to use mental computation and estimation to evaluate calculator and computer results.
- Familiarity with the methods used to solve mathematical problems when calculators or computers are the tools.

From Academic Preparation For College: What Students Need to Know and Be Able to Do. *New York: College Board, 1983. Reprinted with permission.*

Statistics

- The ability to gather and interpret data and to represent them graphically.
- The ability to apply techniques for summarizing data using such statistical concepts as average, median, and mode.
- Familiarity with techniques of statistical reasoning and common misuses of statistics.

Algebra

- Skill in solving equations and inequalities.
- Skill in operations with real numbers.
- Skill in simplifying algebraic expressions, including simple rational and radical expressions.
- Familiarity with permutations, combinations, simple counting problems, and the binomial theorem.

Geometry

- Knowledge of two- and three-dimensional figures and their properties.
- The ability to think of two- and three-dimensional figures in terms of symmetry, congruence, and similarity.
- The ability to use the Pythagorean theorem and special right triangle relationships.
- The ability to draw geometrical figures and use geometrical modes of thinking in the solving of problems.

Functions

- Knowledge of relations, functions, and inverses.
- The ability to graph linear and quadratic functions and use them in the interpretation and solution of problems.

College entrants expecting to major in science or engineering or to take advanced courses in mathematics or computer science will need the following more extensive mathematical proficiency.

Computing

- *The ability to write computer programs to solve a variety of mathematical problems.*
- *Familiarity with the methodology of developing computer programs and with the considerations of design, structure, and style that are an important part of this methodology.*

Statistics

- *Understanding of simulation techniques used to model experimental situations.*
- *Knowledge of elementary concepts of probability needed in the study and understanding of statistics.*

Algebra

- *Skill in solving trigonometric, exponential, and logarithmic equations.*
- *Skill in operations with complex numbers.*
- *Familiarity with arithmetic and geometric series and with proofs by mathematical induction.*
- *Familiarity with simple matrix operations and their relation to systems of linear equations.*

Geometry

- *Appreciation of the role of proofs and axiomatic structure in mathematics and the ability to write proofs.*
- *Knowledge of analytic geometry in the plane.*
- *Knowledge of the conic sections.*
- *Familiarity with vectors and with the use of polar coordinates.*

Functions

- *Knowledge of various types of functions including polynomial, exponential, logarithmic, and circular functions.*
- *The ability to graph such functions and to use them in the solution of problems.*

Recommendations for the Preparation of High School Students for College Mathematics Courses

The Mathematical Association of America
The National Council of Teachers of Mathematics

The following statement, adopted by the Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics, is a brief outline of the basic ingredients of adequate preparation for collegiate level mathematics.* The statement does not break new ground; it reflects standards that have been generally accepted for over a decade. It is intended to support the continuing effort of conscientious teachers everywhere to provide their students with sound and stimulating mathematical training. It is specifically designed to provide a benchmark for our efforts and those of others to assess and react to recent reports of a general decline in the performance of students in mathematics.

A joint committee of the Mathematical Association of America and the National Council of Teachers of Mathematics consulted with secondary school and college teachers in various parts of the country to study recent trends in the preparation of students. The comments from these consultations on which there was strongest consensus are the basis for this statement and its ten recommendations.

The Mathematical Association of America and the National Council of Teachers of Mathematics wish to emphasize that the statement and recommendations, as they refer to secondary school programs, are addressed only to those programs for students planning to go to college and that they are not intended to be more comprehensive. During the past twenty years many important changes have taken place in both the content and teaching of mathematics at the secondary school level. Many excellent new programs were adopted and taught effectively by teachers in elementary and secondary schools. Nevertheless, any consideration of the relative merits of new versus traditional school curricula has been deliberately avoided. A study of this issue would have exceeded both the charge to the committee and its limited resources. This statement and these recommendations incorporate many of the best features of both of these curricula and are addressed to all mathematics programs regardless of pedagogical heritage.

Necessary Course Work

Mathematics is a highly structured subject in which various concepts and techniques are highly dependent upon each other. The concepts of arithmetic and algebra, however, are basic to all of mathematics. Further work in mathematics and in all areas in which mathematics is used as a tool require correct performance, with understanding, of basic arithmetic operations, manipulation of algebraic symbols and understanding of what the manipulations mean.

Any student who is unable to perform arithmetic calculations and algebraic operations with accuracy and reasonable speed, to understand which operations to use in a given problem, and to determine whether the results have meaning is severely handicapped in the study and applications of mathematics. The prevalence of inexpensive pocket calculators make the performance of complicated calculations less tedious, but the use of calculators does not lessen the need for students to understand which concepts and operations are needed to solve a problem, to make sensible estimates and to analyze their results.

For further work in mathematics, and in many other areas, from business to psychology, from biology to engineering, the ability to use algebra with skill and understanding is also essential.

**Collegiate mathematics refers to courses in calculus (or calculus and analytic geometry), probability and statistics, finite mathematics, and higher level mathematics courses.*

Having a passing grade in algebra is not enough. Both understanding and competence in the skills of algebra are necessary. Neither conceptual understanding alone nor technical skill alone will suffice in today's world, let alone in tomorrow's. Algebra is a useful subject which will help to solve problems in the real world. Opportunities to apply algebraic skills should be provided whenever possible, especially to problems that show the utility of mathematics.

Algebra courses in secondary school should include, in addition to the basic topics:

a. polynomial functions,
b. properties of logarithms,
c. exponential and logarithmic functions and equations,
d. arithmetic and geometric sequences and series,
e. the binomial theorem,
f. infinite geometric series,
g. linear and quadratic inequalities.

For most students adequate coverage of the topics in algebra, requires at least two years of study.

Students who will take calculus—and this now includes very many students who will take college work in business, premedicine, economics, biology, statistics, engineering, and physical science—may or may not need trigonometry, depending on the type of calculus course appropriate for their particular programs, but they will need a good deal of what is often call "pre-calculus," including especially a sound understanding of the concept of a function, which is also fundamental for work beyond the most elementary level in probability and computing.

Those students needing trigonometry should study:

a. trigonometric functions and their graphs,
b. degree and radian measure,
c. trigonometric identities and equations,
d. inverse trigonometric functions and their graphs.

For such students, the equivalent of one semester should be devoted to the study of the topics in trigonometry.

All students who go on to take collegiate mathematics will find their college work easier if they have been introduced to some axiomatic system and to deductive reasoning. Traditionally this has been accomplished in a geometry course. Geometry courses in secondary school should include, in addition to basic topics:

a. fundamental properties of geometric figures in three dimensions,
b. applications of formulas for areas and volumes,
c. experience in visualizing three-dimensional figures.

Other courses** beyond algebra, trigonometry, and geometry should be available to students who have the adequate background and the time to take them. A course in coordinate (or analytic) geometry is ideal since it combines algebra with geometry and provides a useful preparation for calculus. In addition to coordinate geometry, courses in the following topics are valuable: probability, statistics, elementary finite mathematics (or linear algebra), an introduction to computers and computing, and applications of mathematics.

If coordinate geometry is offered, it should include in addition to the basic topics:

a. conic sections,
b. rational functions and their graphs,
c. polar coordinates,
d. parametric equations and their graphs.

Inductive as well a deductive reasoning, techniques of estimation and approximation, and an awareness of problem-solving techniques, with special emphasis on the transition from the verbal form to the language of mathematics, should be emphasized in all courses.

Calculus, where offered in secondary schools, should be at least a full year course and be taken only by those students who are strongly prepared in algebra, geometry, trigonometry, and coordinate geometry.

We recognize that many secondary schools have a curriculum similar to that outlined above. We emphasize again that, in order to be properly prepared for collegiate level courses in mathematics, students need to develop skills

i. in applying standard techniques, and
ii. in understanding of important concepts.

***The word "course" refers here and elsewhere in this statement to a semester course unless otherwise noted.*

Recommendations

The Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics make the following recommendations:

1. Proficiency in mathematics cannot be acquired without individual practice. We, therefore, endorse the common practice of making regular assignments to be completed outside of class. We recommend that parents encourage their children to set aside sufficient time each day to complete these assignments and that parents actively support the request of the teachers that homework be turned in. Students should be encouraged to develop good study habits in mathematics courses at all levels and should develop the ability to read mathematics.

2. Homework and drill are very important pedagogical tools used to help the student gain understanding as well as proficiency in the skills of arithmetic and algebra; but students should not be burdened with excessive or meaningless drill. We, therefore, recommend that teachers and authors of textbooks step up their search for interpreting problems that provide the opportunity to apply these skills. We realize that this is a difficult task, but we believe that providing problems that reinforce manipulative skill as a by-product should have high priority, especially those that show mathematics helps solve problems in the real world.

3. We are aware that teachers must struggle to maintain standards of performance in courses at all levels of kindergarten through college and that serious grade inflation has been observed. An apparent growing trend to reward effort or attendance rather than achievement has been making it increasingly difficult for mathematics teachers to maintain standards. We recommend that mathematics departments review evaluation procedures to insure that grades reflect student achievement. Further, we urge administrators to support teachers in this endeavor.

4. In light of 3 above, we also recognize that advancement of students without appropriate achievement has a detrimental effect on the individual student and on the entire class. We, therefore, recommend that school districts make special provisions to assist students when deficiencies are first noted.

5. We recommend that cumulative evaluations be given throughout each course, as well as at its completion, to all students. We believe that the absence of cumulative evaluation promotes short-term learning. We strongly oppose the practice of exempting students from evaluations.

6. We recommend that computers and hand calculators be used in imaginative ways to reinforce learning and to motivate the student as proficiency in mathematics is gained. Calculators should be used to supplement rather than to supplant the study of necessary computational skills.

7. We recommend that colleges and universities administer placement examinations in mathematics prior to final registration to aid students in selecting appropriate college courses.

8. We encourage the continuation or initiation of joint meetings of college and secondary school mathematics instructors and counselor in order to improve communication concerning mathematics prerequisites for careers, preparation of students for collegiate mathematics courses, joint curriculum coordination, remedial programs in schools and colleges, an exchange of successful instructional strategies, planning of inservice programs, and other related topics.

9. Schools should frequently review their mathematics curricula to see that they meet the needs of their students in preparing them for college mathematics. School districts that have not conducted a curriculum analysis recently should do so now, primarily to identify topics in the curriculum which could be either omitted or de-emphasized, if necessary, in order to provide sufficient time for the topics included in the above statement. We suggest that, for example, the following could be de-emphasized or omitted if now in the curriculum:

   a) logarithmic calculations that can better be handled by calculators or computers,
   b) extensive solving of triangles in trigonometry,
   c) proofs of superfluous or trivial theorems in geometry.

10. We recommend that algebraic concepts and skills be incorporated wherever possible in geometry and other courses beyond algebra to help students retain these concepts and skills.

Essential Objectives for College Preparatory Mathematics

UW-System Committee on Mathematics Placement Testing

Success in college level mathematics depends on mastery of many skills and the ability to apply many concepts. The list of objectives which follows identifies those skills and concepts considered essential for potential success in a variety of entry level university courses. Other topics are important but are considered as related to or prerequisite for those listed. Some of these will be addressed following the objectives.

Statewide mathematics placement testing is referenced to this list of objectives.

The objectives are listed in the following order:

Objective	**Typically**
1. Pre-algebra	(pre-9th grade)
2. Elementary algebra	(9th grade)
3. Advanced algebra	(11th grade)
4. Geometry	(10th grade)
5. Pre-calculus	(12th grade)

If a student has mastered the skills associated with the objectives in 1 through 5, he/she will usually be placed in a Calculus I course. If a student has mastered the skills associated with the objectives 1 through 4, he/she will usually be placed in a pre-calculus course or a short course in calculus (for non-science, non-engineering students). Other students will be placed in background courses and may experience delays in getting into some degree programs.

Essential Objectives

It cannot be emphasized too strongly that what follows is a set of objectives for a placement test rather than a description of a high school mathematics education. We have taken the view that a placement test should cover only topics and skills which are pre-requisite to some entry level college course in the sense of being used in the course without being taught in that course. In consequence, some areas of mathematics are less fully represented in our objectives than they should be in high school courses, and other areas are omitted entirely even though they are both suitable and valuable for high school mathematics.

Even among pre-requisite skills, not all are represented in the objectives because not all fit the placement test format in which we were working. This format requires broad area coverage in a short time span and machine gradability. The practical effect of these constraints is to rule out problems having any substantial degree of complexity. This, in turn, rules out of our list those objectives which can be adequately tested only by use of complex problems—no matter how important such objectives may be for college work.

Although we have listed these objectives in a definite progression, we do not mean that every mathematical idea will be tested only at that level in the progression. The rule is that a concept mentioned at one point in this list may be used as part of all subsequent objectives on the list, or may occur on the test in a form more complicated than suggested by the level at which the concept is mentioned.

1. Pre-Algebra

Students should be able to:

1.01 Obtain quotient and remainder given a two-digit divisor and a three-digit divisor.

1.02 Estimate the quotient for a division problem.

1.03 Check division through the use of multiplication.

1.04 Perform basic arithmetic operations with, and reduce to lowest terms, fractions involving single-digit whole numbers for numerator and denominator.

1.05 Perform basic arithmetic operations with mixed numbers.

1.06 Perform basic arithmetic operations, properly aligning the decimal point, with decimal numbers.
1.07 Convert between fractions, decimals, and percents.
1.08 Arrange fractions, decimals, and integers in order of size.
1.09 Set up and solve simple verbal problems involving percentages, mixtures, unit conversions, and distance/time/rate.
1.10 Find areas of geometric figures composed of rectangles and triangles.
1.11 Find circumferences and areas of circles.
1.12 Find the surface area and volume of a rectangular prism.
1.13 Solve problems involving the measures of angles of triangles, including using the fact that the sum of the measures of angles of any triangle, is 180°.
1.14 Exhibit knowledge of basic geometric vocabulary.
1.15 Find areas and perimeters of basic plane geometric figures.

2. Elementary Algebra

Students should be able to:

2.01 Perform the basic arithmetic operations with integers, using symbols of grouping.
2.02 Solve linear equations containing simple fractions and literal numbers.
2.03 Manipulate formulas containing fractions and several variables.
2.04 Simplify algebraic expressions involving multiplication and division of exponential expressions.
2.05 Simplify exponential expressions with negative bases.
2.06 Find products of binomials.
2.07 Collect like terms, and perform simple monomial factoring.
2.08 Use order of operations to simplify polynomials and rational expressions.
2.09 Evaluate polynomials at specific values of the variable.
2.10 Set up and solve verbal problems involving linear equations.

3. Advanced Algebra

Students should be able to:

3.01 Simplify (reduce) rational expressions.
3.02 Add, subtract, multiply, and divide rational expressions.
3.03 Factor into linear factors a quadratic trinomial in one or two variables.
3.04 Recognize the relationships between coefficients of perfect-square trinomials.
3.05 Factor perfect-square trinomials.
3.06 Evaluate numerical expressions containing integral and fractional exponents.
3.07 Simplify radical expressions.
3.08 Rationalize numerators and denominators of rational expressions containing radicals.
3.09 Rewrite radical expressions using fractional exponents.
3.10 Rewrite expressions containing fractional and integral exponents into expressions containing radicals.
3.11 Add, subtract, multiply, and divide radical expressions.
3.12 Simplify complicated radical expressions.
3.13 Simplify algebraic expressions with positive, negative, and zero exponents.
3.14 Convert between decimal and scientific notation.
3.15 Graph a straight line given its equation.
3.16 Find the slope and y-intercept of a straight line given its equation.
3.17 Find the equation of a line given two points.
3.18 Find the equation of a line given its slope and one point.
3.19 Graph a parabola given its equation.
3.20 Evaluate numerical expressions involving absolute values.
3.21 Recognize the graph of $y = |x|$.
3.22 Solve equations involving absolute values.
2.23 Graph the solution of $Ax + B < 0$, ≤ 0, > 0, ≥ 0 on the number line.
3.24 Graph the solution of $Ax + By + C < 0$, ≤ 0, > 0, ≥ 0 in the coordinate plane.

3.25 Solve quadratic equations (includes use of quadratic formula).

3.26 Solve fractional equations, discarding extraneous roots.

3.27 Solve a system of two linear equations in two unknowns by substitution and/or addition; recognizing cases of no solutions or infinitely many solutions.

3.28 Graph a system of two linear equations in two unknowns.

3.29 Find special products of squares and cubes of binomials.

3.30 Factor perfect squares, difference of two squares, sum and difference of two cubes.

3.31 Read, analyze, and solve verbal problems.

4. Geometry

Students should be able to:

4.01 Recognize and use congruence and similarity correspondences.

4.02 Recognize and use conditions which imply congruence and those which imply similarity for triangles.

4.03 Recognize and use symmetries of isosceles triangles and equilateral triangles.

4.04 Set up and solve proportions derived from similar triangles.

4.05 Recognize and use instances where perpendicular lines and segments occur, e.g. tangent to a circle is perpendicular to radius at point of tangency; angle inscribed in semi-circle is a right angle; if measures of sides of triangle satisfy $a^2 + b^2 = c^2$, the angle opposite side of length c is a right angle; bisector of angle opposite base in isosceles triangle is perpendicular to base; diagonals of a rhombus are perpendicular.

4.06 Recognize and use conditions which imply that two lines are parallel or two lines are perpendicular.

4.07 Recognize and use instances where parallel lines and segments occur, e.g. two coplanar lines perpendicular to a third line are parallel to each other; a segment whose endpoints divide two sides of a triangle in the same proportion is parallel to the third side of the triangle; similar polygons may be oriented so that corresponding sides are parallel.

4.08 Find the area of a sector of a circle given the measure of a central angle.

4.09 Find measures of angles by using complementary and supplementary relations.

4.10 Use Pythagorean relationship to find length of a side of a right triangle.

4.11 Recognize and use the fact that the contrapositive of a conditional statement has the same truth value as the conditional statement.

4.12 Given a generalization, identify examples and counterexamples; e.g., recognize that skew lines are counterexamples to the statement that all lines which do not intersect are parallel.

4.13 Classify sets of objects by set inclusion, e.g. the set of squares is a subset of the set of rectangles.

5. Precalculus

Students should be able to:

5.01 Use the definitions of the trigonometric functions of angles.

5.02 Evaluate trigonometric functions for special angles; e.g.,

$$30°,\ \frac{\pi}{3},\ \frac{5\pi}{4},\ \textit{etc.}$$

5.03 Use the standard formulas and identities: Pythagorean formulas, quotient and reciprocal formulas, complementary angle formulas, sums and difference formulas, and double- and half-angle formulas.

5.04 Solve trigonometric equations (linear and quadratic).

5.05 Sketch the graphs of trigonometric functions.

5.06 Evaluate expressions involving inverse trigonometric functions.

5.07 Solve right triangles for angles and sides and find areas.

5.08 Solve obtuse and acute triangles for angles and sides, and find areas.

5.09 Evaluate functions for numerical and literal arguments.

5.10 Determine the domain and range of a function from a defining statement or graph.
5.11 Add, multiply, and divide functions.
5.12 Find difference quotients.
5.13 Find the composition of two functions.
5.14 Work with exponential and logarithm functions and their graphs.
5.15 Solve exponential and logarithm functions and their graphs.
5.16 Solve inequalities of the types $|ax + b| < c$, $|ax + b| \leq c$, $|ax + b| > c$, $|ax + b| \geq c$, and graph the solutions.
5.17 Solve quadratic inequalities in one variable, and graph the solutions.
5.18 Do the arithmetic of complex numbers.
5.19 Represent complex numbers geometrically.
5.20 Use remainder and factor theorems to find the zeros of polynomials.
5.21 Solve systems containing one linear and one quadratic equation in two variables.
5.22 Solve systems containing two quadratic equations in two variables.
5.23 Find the distance between two points.
5.24 Determine from the equations of two lines whether they are parallel or perpendicular or neither.
5.25 Convert between the equation of a circle in standard form and data on the center and radius.
5.26 Convert between the equation of a parabola in standard form and data on its vertex and axis.
5.27 Distinguish equations of conics.

Additional Statements about High School Preparation for College Mathematics Study

Geometry

It should be apparent that the range of objectives included represents a small portion of the objectives of the traditional high school geometry course. It is also apparent that the algebra objectives in the working document represent a substantial portion of the objectives of traditional high school algebra courses. The imbalance of test objectives can be explained in part by the nature of the entry level mathematics courses available at most colleges.

The first college mathematics course will almost always be either calculus or some level of algebra. The most appropriate course for an individual is a local placement decision usually based on three factors: (1) high school background; (2) placement test score or scores; (3) curricular objectives. One reason for the emphasis on algebra in the working document is that virtually all college placement decisions involve a placement into a course which is more algebraic than geometric in character. Experience has demonstrated that placement tests which emphasize algebra are more reliable and consistent than tests which include more non-algebra items.

It has been conjectured that a recognition of (1) a lack of emphasis on geometry in college placement tests and (2) a lack of geometry courses at the entry level in college would lead to questions about the value of the high school geometry course. The testing committee does not believe that placement test items or objectives should serve as goals for high school geometry. Committee members were asked this question: If a college-bound student were to take two years of high school mathematics, would you recommend two years of algebra or one year of algebra and one year of geometry? Naturally, committee members prefer that college-bound students take more than two years of mathematics in high school. None, however, would recommend two years of algebra over one year of algebra and one year of geometry. That collective opinion suggests that the high school geometry course has an important role in a college preparatory program.

There are several reasons for maintaining a geometry course as an essential component in a college preparatory program. Three will be given here. (1) Since there are no entry level courses in geometry at the college level, it is essential that students accomplish basic geometry objectives while in high school. (2) Geometry plays a much more significant role in college entry level courses than is suggested by the placement test items and objectives. High school geometry can contribute to a level of mathematical maturity which is important for success in college but which cannot be adequately described using specific behavioral objectives.

(3) That proficiency in algebra may be a valid predictor of success in college mathematics does not mean that proficiency in algebra is sufficient for success in college mathematics. One who would guarantee success would advise students to expose themselves to a wide range of objectives, including those in the area of geometry, rather than concentrate on a single area.

Probability and Statistics

Although university curricula are somewhat in a state of flux, with many basic issues and philosophies being examined, the normal entry level courses in mathematics remain the traditional algebra and calculus courses. Therefore, the placement exams must reflect those skills which would be necessary for success in these courses. This is not intended to imply that courses stressing topics other than algebra and geometry are not vital to the mathematics curriculum but rather that those topics do not assist in placing students in the traditional university entry level path.

A major example of courses that are of extreme value in the mathematical training of young people today but are not reflected on the placement exam are the topics of probability and statistics. It is the committee's feeling that these topics are vital to the elementary and secondary curriculum. They are indeed growing in importance on university campuses, both within mathematics departments and within those departments not normally though of as being quantitative in nature. As the social sciences mature, they are seeking mathematical models to apply, and in general these models tend to be probabilistic or statistical in nature. As a result, the curriculum in these areas is becoming heavily permeated with probabilistic and statistical topics.

Mathematics departments are finding that their graduates are going, in large part, into jobs utilizing backgrounds in computer sciences or statistics, and consequently, their curricula are beginning to reflect these trends. It is possible that in the near future there will be a variety of entry level courses in mathematics requiring placement relative to knowledge in statistics. Until such time, however, the committee urges the education community to develop and maintain meaningful instruction in probability and statistics.

Logic

Two points should be made under this heading. First, we have presented our objectives in a form which suggests that they are a collection of unrelated facts when they are really part of a logically related structure. If mathematics may be likened to a cathedral, then our list of objectives is a catalog of the stones out of which the cathedral is built. A well-prepared student should have some sense of what the structure looks like.

Second, there are parts of logic of which entering college students should have a working command. We emphasize that what is needed is the ability to use these parts as they arise in a mathematical context, and not the ability to give formal statements in a logical context. These parts include:

1. Correct use of "and" and "or" as connectives between statements, and to translate to intersections and unions of sets.
2. Ability to interpret statements of the form "if A, then B."
    a) When given such a statement, to check whether A holds, and if so, to use B; and if not, to refrain from using B.
    b) To refrain from checking whether B holds, and if so, then using A.
3. Realizing that a statement about all elements in a set cannot be established by checking specific examples (unless every point in the set is checked), but that it can be refuted by a simple example where it fails to hold. Being prepared to try specific examples of a general statement as a means of conjecturing whether it holds.
4. Realization that two sets with different definitions may or may not be disjoint, and may or may not together exhaust the universe of definition.

Problem Solving

By "problem solving" we mean the ability to select and combine individual mathematical operations in whatever way is necessary to solve a problem of some complexity. Ideally, a set of problem-solving skills would appear in the list of objectives since such skills (at an appropriate level) are prerequisite to entry-level college courses. In fact, problem solving does not appear in the objective list because of limitations of the conventional placement test format. This format emphasizes brevity and conceptual simplicity of problems, and these qualities are antithetical to testing problem-solving skills. In short, the omission of problem solving from the objectives is a good example of the fact that they are objectives for a placement test not for a mathematics education.

Although the essence of problem solving is the ability to combine component skills, one can list some of the elements of problem solving separately.

Such a list would include the following:

1. Identify the quantity or relation to be found or constructed.

2. Assign variables to unknown quantities, keeping track of units and making conversions as necessary for consistency.

3. Translate the data of the problem into appropriate mathematical statements (functions, relations, equations).

4. Divide a complex problem into tractable subproblems, identifying which data, variables, etc. go with each subproblem.

    Examples of this include:

    —To find the distance from a given point P to a given line L.

    a) Find the equation of the line through P perpendicular to L.
    b) Find the intersection Q of this line with L.
    c) Compute the distance PQ.

    —To solve a problem containing an unknown parameter.

    a) Use one set of data to obtain a value for the parameter.
    b) Use the value obtained plus additional data to obtain the desired solution.

    One of the key abilities at this step is the ability to recognize which subproblems are solvable before actually solving them.

5. Performing the necessary operations for solving each subproblem.

6. Combine solutions to subproblems (undoing any substitutions made along the way) to obtain the desired result.

7. Check solution(s) in the original statement of the problem and discard extraneous ones.

A problem requiring all the above steps for its solution would, of course, be at the level which goes with the C component of the placement test. Problems at the level of the A component of the test would be solvable by use of only a single variable and without any formal division into subproblems.

Calculus

In judging the impact of high school calculus courses, it is necessary to keep in mind three categories of students. The first category consists of students who learn enough there to skip a semester of calculus in college. For these, the high school course is clearly beneficial. The second category consists of students who learn somewhat less and have to begin their college math with first semester calculus. For them, exposure to high school calculus is not always beneficial. It has, understandably, led them to think of themselves as somewhat of a mathematical elite; it has also taught them well the beginning of the college calculus course. These two factors lead a number of these students to be inattentive in their college course until the moment an exam demonstrates that the course has moved beyond the material they have mastered. Discovering areas of ignorance on an exam is, by definition, discovering them too late. Finally, there are students whose high school pre-calculus has been abridged in the interests of getting in a high school calculus course and who, as a result, have to start college math with a pre-calculus course. For such students, the high school calculus course has been a clear loss.

We have no wisdom to offer on how to balance the claims of students in these three categories, but we do suggest that high schools offering a calculus course attempt to find out what fraction of its graduates fall into each category.

Calculus topics are omitted from the objectives for a simpler reason: Courses having calculus skills as a pre-requisite are not considered "entry-level" and are by definition outside the scope of our tests. However, most UW campuses do have arrangements whereby incoming students can demonstrate calculus skills and, if their skills are sufficient, start in higher level courses and/or secure calculus credit.

Applications of Mathematics in Other Disciplines

Quantitative science courses are very valuable ways for students to deepen and consolidate their mathematical understanding. While there exist non-quantitative course called "chemistry" or "physics," these two areas are the most likely to be quantitative, and students seeking well-developed quantitative skills are encouraged to take one or both.

Applications should also be included in math courses themselves. However, the purely mathematical skills listed in the objectives are the most central for college preparation, so college-bound students will be ill-served if any of these are neglected in the pursuit of applications.

Computing Courses

The remarks made under "Applications" apply here as well, including the fact that a course can deal with computers without in any way dealing with mathematics. Quite aside from the issue of mathematical preparation, almost everyone who aspires to be educated at the college level should, at some stage, learn to communicate with a computer.

The Nature of Mathematics

One of the properties of mathematics that does not lend itself to evaluation by written examinations is a rather elusive one which for want of more precise name we can label "the nature of Mathematics."

When people are asked to respond to the question "What is mathematics?" their response usually deals with mechanical operations on number, geometric entities such as area, volume; in short on that aspect of mathematics directly related to everyday human affairs. After all this is the mathematics common to the educational experiences of all people who have completed at least 10 years of school. Completely omitted in an answer such as this is the fact that in mathematics and its development we also have a unique, powerful and fascinating way of thought. It involves induction, deduction, trial and error, guessing, intuition, all of the things which illustrate the power of human reason. Although this process of though should pervade the study of mathematics and may aid the student in responding to questions such as those on the placement test, it is not tested directly.

A Three Year Curriculum for Non-College-Bound Students

Course Content: Year One

Arithmetic

1. Recognize many different uses of numbers.
    a. Distinguish among numbers used as counts, measures, locations in reference frameworks, scales, codes, names.
    b. Recognize the need for counting numbers, rational numbers, negative numbers, and other sets of numbers.
    c. Use numbers as locators.
    d. Interpret formulas as ways of expressing numbers.

2. Associate arithmetic operations with physical world phenomena.
    a. Associate addition with putting together and shifting.
    b. Associate subtraction with taking apart, shifting, and comparison.
    c. Associate multiplication with a number of N-tuples, size change, area and volume, acting through, and rate factor.
    d. Associate division with rate comparisons, ratio comparisons, and recovering factors from products.

3. Express numbers in equivalent and approximate forms.
    a. Order whole numbers and decimals.
    b. Recognize many names for a given number, including equivalent fractions.
    c. Round numbers to an indicated place value.
    d. Rewrite numbers in different forms.

4. Use approximation to estimate the results of computations with whole numbers, fractions, and decimals.
    a. Estimate the results of a computation to one significant digit.

5. Use a calculator to compute with whole numbers, integers, and decimals, and use mental arithmetic or pencil-and-paper algorithms where appropriate.
    a. Read and interpret parentheses as grouping symbols.
    b. Apply conventions regarding order of operations.
    c. Solve formulas in which the unknown variable stands alone on one side of the equation.
    d. Use standard algorithms to add, subtract, and multiply two-digit numbers and divide a three-digit number by a one-digit number or multiple of ten.
    e. Use a calculator to compute with whole numbers and decimals.

6. Solve routine problems.
    a. Solve one-step word problems using a calculator where appropriate.

Algebra

1. Use variables and the language of algebra.
    a. Use letters to name numbers.
    b. Use expressions including letters to describe a sequence of operations involving an unknown number.
    c. Interpret parentheses and other grouping symbols to denote order of operations.
    d. Use conventions regarding order of operations.
    e. Use 2 and 3 as exponents for numbers and variables and $b = \sqrt{x}$ as a solution for $b^2 = x$.

2. Find an unknown value using a formula.
    a. Substitute known values into a formula.
    b. Solve an equation for an unknown value that appears alone on one side of the equation.

3. Simplify elementary algebraic expressions.
    a. Solve proportions as equal ratios or equivalent fractions.

   b. Recognize the equivalence of $ac + ad$ and $a(c + d)$.
4. Solve linear and quadratic equations.
   a. Use guess-and-test procedures to solve linear equations in one variable.
   b. Recognize the equivalence of $a + x = b$ and $x = b - a$.
   c. Recognize the equivalence of $bx = a$ and $x = a/b$.
5. Associate equations with their graphs.
   a. Plot points in a Cartesian coordinate system.
6. Represent problem situations by open sentences.
   a. Write equations for one-step routine problems.

Geometry

1. Recognize and name common geometric figures.
   a. Recognize angles, triangles, squares, rectangles, parallelograms, rhombuses, trapezoids, circles, prisms, cylinders, and spheres, and use the terminology associated with each.
2. Identify properties of common geometric figures.
   a. Describe any figure listed in item 1a in such a way as to distinguish it from any other.
3. Determine measurement properties of common geometric figures.
   a. Use instruments appropriate for measuring angles, length, area, and capacity.
   b. Know and use the following angle measurement properties of triangles: the sum of the measures of the angles of a triangle is 180 degrees, the base angles of an isosceles triangle are congruent, the line of symmetry of an isosceles triangle bisects the vertex angle and is perpendicular to the base.
   c. Know and use the following segment measure properties of triangles: the Pythagorean theorem, 30-60-90 and 45-45-90 triangle relationships, the line of symmetry of an isosceles triangle bisects the base.
   d. Find perimeter and area of triangles and rectangles, and volume of right prisms.
4. Recognize relations among common geometric figures.
   a. Identify right angles, supplementary angles, vertical angles, alternate interior angles, and corresponding angles.
   b. Identify parallel and perpendicular segments, rays, and lines.
5. Apply properties of figures and relations among figures to determine unknown measures or properties.
   a. Use the measurement properties in item 3b and the measurement properties of those figures listed in 4a to determine the measure of angles.
6. Locate points and lines in one-, two-, or three-dimensional coordinate systems.
   a. In one- and two-dimensional coordinate systems, locate a point whose coordinates are given or give the coordinates of a specified point.

Statistics and Probability

1. Collect and organize data into tables, charts, and graphs.
   a. Identify procedures for collecting data for an experiment.
   b. Collect data through surveys, physical experiments, and other sources.
   c. Organize data into graphs, tables, and charts.
   d. Organize data using stem-and-leaf techniques.
   e. Present data to others in a clear form.
2. Read and interpret tables, charts, and graphs.
   a. Identify different uses of numbers in statistical reporting.
3. Use central tendency measures and dispersion measures.
   a. Identify different uses of numbers in statistical reporting.
4. Understand simple probability statements.
   a. Use counting skills to determine probabilities as statements of relative frequency for independent events.
5. Use statistics in decision making.

Course Content: Year Two

Arithmetic

1. Recognize many different uses of numbers.
   a. Interpret numbers used as scales.

   - b. Use numbers to interpret a probability statement.

2. Associate arithmetic operations with physical world phenomena.
   (Review and reinforce previous objectives.)
3. Express numbers in equivalent and approximate forms.
   - a. Order fractions and decimals.
   - b. Express whole numbers and decimals in scientific notation.
   - c. Express fractions as equivalent decimals and decimals as equivalent fractions.
   - d. Express fractions and decimals as equivalent percents and percents as equivalent fractions or decimals.
   - e. Use 2 and 3 as exponents and interpret square root and cube root as their inverses.
4. Use approximation to estimate the results of computations with whole numbers, fractions, and decimals.
   - a. Estimate results of a computation to two significant digits.
5. Use a calculator to compute with whole numbers, integers, and decimals, and use mental arithmetic or pencil-and-paper algorithms where appropriate.
   - a. Add, subtract, and multiply fractions having denominators of 2, 3, 4, 8, and 10.
6. Solve routine problems.
   - a. Solve one-step problems involving fractions and decimals.
   - b. Solve rate problems using diagrams illustrating the many-to-many correcpondence.
   - c. Solve multiple-step problems.

Algebra

1. Use variables and the language of algebra.
   (Review and reinforce previous objectives.)
2. Find an unknown value using a formula.
   (Review and reinforce previous objectives.)
3. Simplify elementary algebraic expressions.
   - a. Add, subtract, multiply, and divide monomials.
   - b. Recognize the equivalence of $\frac{a\,b}{a\,c}$ and $\frac{b}{c}$ when $a \neq 0$.
   - c. Recognize the equivalence of $\frac{a}{b} + \frac{c}{d}$ and $\frac{ad + bc}{bd}$
4. Solve linear and quadratic equations.
   - a. Solve linear equations in one variable.
   - b. Recognize the equivalence of $ab = 0$ and $a = 0$ or $b = 0$.
   - c. Solve quadratic equations in one variable using the quadratic formula (introduced without proof but verified by examples).
   - d. Solve $\frac{a}{b} = \frac{c}{d}$ for $a, b, c,$ or d.
5. Associate equations with their graphs.
   - a. Construct the graph of a linear equation in two variables.
   - b. Determine the slope, x-intercept and y-intercept of a linear equation in x and y.
6. Represent problem situations by open sentences.
   - a. Write equations for multiple-step routine problems.
   - b. Write equations for routine rate problems.

Geometry

1. Recognize and name common geometric figures.
   - a. Recognize pyramids and cones and use the terminology associated with each.
2. Identify properties of common geometric figures.
   - a. Identify squares as rectangles; rectangles and rhombuses as parallelograms.
   - b. Know and use the common properties of rectangles and parallelograms.
3. Determine measurement properties of common geometric figures.
   - a. Find approximate lengths and areas of irregular figures.
   - b. Determine that C/d is constant for all circles.
   - c. Know and use area and circumference relationships for circles.
   - d. Find area of parallelograms.
   - e. Find volume of rectangular and triangular prisms and right circular cylinders.
4. Recognize relations among common geometric figures.
   - a. Identify parallel and perpendicular planes.

   b. Identify congruent and similar triangles.
   c. Identify corresponding parts of congruent and similar triangles.

5. Apply properties of figures and relations among figures to determine unknown measures or properties.
   a. Determine measures of unknown sides and angles in similar or congruent triangles.
   b. Determine measures of central angles or inscribed angles when the arc is given, or of the arc when the central angle or the inscribed angle is given.

6. Locate points and lines in one-, two-, or three-dimensional coordinate systems.
   a. Determine the graph of a linear equation in two variables.
   b. Determine the equation of a line in a two-dimensional coordinate system.
   c. In a three-dimensional coordinate system, locate a point whose coordinates are given or give the coordinates of a specified point.

Statistics

1. Collect and organize data into tables, charts, and graphs.
   (Review and reinforce previous objectives.)
2. Read and interpret tables, charts, and graphs.
   a. Interpret tables, charts, and graphs.
3. Use central tendency measures and dispersion measures.
   a. Determine mean, median, and mode for a set of data.
   b. Construct box-and-whisker plots as a way of showing dispersion of data in a frequency distribution.
   c. Calculate and use standard deviation for a set of data.
4. Understand simple probability statements.
   a. Construct and use sample spaces and tree diagrams as counting techniques in determining probabilities.
5. Use statistics in decision making.
   a. Identify elementary misuses of statistics (such as misleading graphs and the use of an inappropriate measure of central tendency).
   b. Identify common frequency distributions (such as rectangular, skewed, J-shaped, bimodal, bell-shaped).
   c. Use statistical techniques and measures to solve appropriate routine and nonroutine problems.

Course Content: Year Three

Arithmetic

1. Recognize many different uses of numbers.
   (Review and reinforce previous objectives.)
2. Associate arithmetic operations with physical world phenomena.
   (Review and reinforce previous objectives.)
3. Express numbers in equivalent and approximate forms.
   a. Approximate fractions by more common fractions.
4. Use approximation to estimate the results of computations with whole numbers, fractions, and decimals.
   (Review and reinforce previous objectives.)
5. Use a calculator to compute with whole numbers, integers, and decimals, and use mental arithmetic or pencil-and-paper algorithms where appropriate.
   a. Add, subtract, and multiply mixed numerals.
   b. Compute with numbers expressed in scientific notation.
6. Solve routine problems.
   (Review and reinforce previous objectives.)

Algebra

1. Use variables and the language of algebra.
   a. Use subscripted variables to represent numbers.
2. Find an unknown value using a formula.
   a. Solve a formula for an unknown value which does not appear alone on one side of the equation.
   b. Evaluate polynomials in x for various values of x.
3. Simplify elementary algebraic expressions.
   (Review and reinforce previous objectives.)
4. Solve linear and quadratic equations.
   a. Solve systems of two equations in two variables by graphing.

   - b. Solve systems of two equations in two variables by algebraic methods.
   - c. Solve inequalities of the form $ax + b < c$.
   - d. Solve equations involving sin, cos, and tan.
5. Associate equations with their graphs.
   - a. Determine the appropriate graph for a linear inequality.
   - b. Determine the equation of a straight-line graph.
   - c. Sketch the graph of a quadratic equation in one variable by plotting points.
6. Represent problem situations by open sentences.
   - a. Write inequality statements representing one-step, multiple-step, and rate problems.

Geometry

1. Recognize and name common geometric figures. (Review and reinforce previous objectives.)
2. Identify properties of common geometric figures.
   - a. Know and use the properties of common three-dimensional figures.
3. Determine measurement properties of common geometric figures.
   - a. Use trigonometric ratios to determine unknown sides or angles in right triangles.
   - b. Find areas of trapezoids.
   - c. Find surface area of rectangular and triangular prisms and right circular cylinders.
4. Recognize relations among common geometric figures.
   - a. Identify congruent and similar figures in two and three dimensions.
5. Apply properties of figures and relations among figures to determine unknown measures or properties.
   - a. Determine measures of unknown sides and angles in similar or congruent figures.
6. Locate points and lines in one-, two-, or three-dimensional coordinate system.

Statistics

1. Collect and organize data into tables, charts, and graphs.
   - a. Construct scatter diagrams.
   - b. Identify basic sampling procedures.
2. Read and interpret tables, charts, and graphs.
   - a. Interpret scatter diagrams as naive measures of correlation.
3. Use central tendency measures and dispersion measures.
   (Review and reinforce previous objectives.)
4. Understand simple probability statements.
   - a. Determine the probability of a combination of independent events.
5. Use statistics in decision making.
   - a. Interpret correlation statements for correlation values between 1 and -1.
   - b. Identify the distinction between correlation and causality.
   - c. Construct null and alternative hypotheses for a hypothesis-testing experiment.
   - d. State the possible consequences of rejection or acceptance of the null hypothesis in a hypothesis-testing experiment.
   - e. Carry out the hypothesis-testing experiment in an applied setting.

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Mathematics Competencies

These competencies were identified in 1984 by a committee of Wisconsin mathematics educators for the first state developed mathematics competency test to be administered under Chapter 241, Laws of 1981, during the 1985-86 academic year. These competencies may be modified for testing in subsequent years.

Grade 3

3.1 The student will be able to recognize pictorial, numerical, and word representations of whole numbers and fractions.

3.2 The student will be able to add and subtract whole numbers.

3.3 The student will be able to determine solutions for number sentences.

3.4 The student will be able to identify squares, rectangles, triangles, and circles.

3.5 The student will be able to use drawings, tables, or graphs to represent or solve problems.

3.6 The student will be able to use addition and subtraction to solve problems in applied situations.

3.7 The student will be able to identify wanted, given, and needed information in a problem statement.

3.8 The student will be able to read instruments for measuring length, time, temperature, capacity, and weight.

3.9 The student will be able to identify appropriate units for measuring length and time.

3.10 The student will be able to recognize equivalent measures and use them to add and subtract measures of length, time, and money.

Grade 7

7.1 The student will be able to recognize equivalent representations of whole numbers and decimals expressed in words, in conventional notation, in expanded notation, or in exponential notation.

7.2 The student will be able to add, subtract, multiply, and divide whole numbers and decimals.

7.3 The student will be able to determine solutions for number sentences including equalities and formulas.

7.4 The student will be able to identify common geometric figures and relationships.

7.5 The student will be able to read and interpret tables and graphs.

7.6 The student will be able to use addition, subtraction, multiplication, and division to solve problems in applied situations.

7.7 The student will be able to solve problems that can be solved by procedures such as looking for a pattern, constructing a table, using guess and check procedures, or using a drawing.

7.8 The student will be able to determine the length, area, volume, and angle measure of common geometric figures.

7.9 The student will be able to round numbers and to estimate the results of addition, subtraction, multiplication, and division of whole numbers, decimals, and fractions.

7.10 The student will be able to recognize equivalent representations of rational numbers expressed as fractions, decimals, or percents.

7.11 The student will be able to add, subtract, and multiply fractions and mixed numbers.

7.12 The student will be able to determine the probability of a simple event.

Grade 10

10.1 The student will be able to recognize equivalent representations of whole numbers, decimals, and fractions expressed in conventional notation, in exponential notation, or as percent.

10.2 (Objective 10.2 was deleted as a result of pilot testing. A new objective will be developed at some future time.)

10.3 The student will be able to evaluate formulas, and to solve proportions, linear equations and linear inequalities with one variable.

10.4 The student will be able to identify similar and congruent figures and determine measures based on similarity and congruence.

10.5 The student will be able to read, interpret, and make inferences from tables and graphs.

10.6 The student will be able to use addition, subtraction, multiplication, and division to solve problems in applied situations.

10.7 The student will be able to solve problems that can be solved by procedures such as looking for a pattern, constructing a table, using guess and check procedures, or making a drawing.

10.8 The student will be able to calculate the length, perimeter, area, volume, or angle measure of common geometric figures.

10.9 The student will be able to order a given set of real numbers.

10.10 The student will be able to determine the outcome of an algorithm (e.g., set of instructions, flow chart, and program) or modify an algorithm to produce a specified outcome.

10.11 The student will be able to solve problems in applied situations involving percent.

10.12 The student will be able to determine the probability of a simple event and to use elementary statistics to describe data.

Girls, Math, and Science

Center for Early Adolescence
University of North Carolina at Chapel Hill

The difference in girls' and boys' achievement in mathematics and science is a topic that has received a great deal of attention in both the popular and academic presses during the last decade. There is a wide spectrum of opinion as to whether differences in achievement are related to gender at all, or whether other phenomena account for the variance between the sexes. One fact seems indisputable; boys and girls achieve at relatively similar rates through the fifth grade level. Marked differences in achievement begin to appear during the middle grades, and these increase year by year (Maccoby & Jacklin, 1974).

Do Boys and Girls Differ in Achievement or Ability?

Few have tried to dispute the basic fact of discrepancy in the math achievement of boys and girls. The larger question of whether the discrepancy in achievement reflects an underlying difference in math ability is, however, much more controversial. In the late 1970s, the National Institute of Education funded a grants competition on the following research question: What are the major positive and negative factors in the study of mathematics that affect female participation, achievement, and career choice? The consensus of the large body of research emanating from the NIE grants was that the primary explanation for sex differences in math achievement is the difference in math course participation. Indeed, as Fennema (1977) pointed out, "What was being compared were not females and males but students who had studied mathematics 1-3 years in high school with students who had studied mathematics 2-4 years in high school" (p. 85).

Math/Science Is Key to Students' Futures

According to Sells (1974), competence in mathematics functions as a "critical filter" in society. The student who does not attain mastery of math concepts defaults on the chance for high-paying technological employment. College-bound students need a full four years of high school mathematics to survive the standard college calculus sequence, which is in turn a prerequisite for many college majors.

Courses in advanced mathematics have become a requirement for all undergraduates, even liberal arts majors, at many universities. Further, advancement even in the so-called "feminine" careers now depends on mathematical competence. For example, the social sciences, including education, require statistics and research design courses in higher degree programs. So does library science, which also demands computer skills. Thus, quantitative skill has become an element of what had previously appeared to be nonquantitative careers.

Students who select themselves out of math courses are thus closing off a wide range of future career and life options. Unfortunately, many girls, even those who have been identified as highly able, drop high school mathematics as soon as the courses become optional. One result has been under representation of females in the most highly paid sectors of the labor force.

Considerably more attention has been paid to sex-related differences in math achievement than in science. To a large extent, according to Kavrell and Petersen (in press), "sex differences in quantitative ability are influential in the development of sex-related differences in science" (p. 2). Thus the math filter serves to "eliminate women from many fields, from chemistry, physics, and engineering, to architecture and medicine" (Ernest, 1976, p. 3).

Computers are increasingly essential in math and science coursework, as well as in the social sciences and the humanities. Since computer science is a relatively new field, sex-role stereotyping has not had the opportunity to become deeply ingrained. Nonetheless, evidence of a growing perception of the computer as a male machine is rapidly beginning to mount (Lockheed & Frakt, 1984). The obvious relationship between computer competence and employability makes early intervention on behalf of girls particularly necessary, in order to give them direct computer experience. Kiesler, Sproull, and Eccles (1983) have pointed out that "girls like to use the computer, but not if they have to fight with boys to get a turn" (p. 48). On the other hand, when teachers "intervened and set up time-sharing rules, the girls spent as much time on the computer as the boys."

The middle-grade years are particularly significant to science and mathematics studies, because they are the last years in which girls and boys retain similar attitudes toward science. In a 1981 study, Haertel, Walberg, Junker, and Pascarella concluded the following:

> Since equality of educational opportunity and the fulfillment of human potential are important, both for society and individuals, it appears that the early adolescent years are formative of important traits related to science learning and attitudes, and are thus an appropriate stage for focusing educational efforts to bring about the greater attainment of these goals. (p.339)

Intervention Essential During Middle-Grade Years

In response to a questionnaire, approximately one-third of the members of the Association for Women in Mathematics reported that they were interested in mathematics by age 11. "Most of the others became interested early in secondary school" (Luchins & Luchins, 1980, p. 10).

Even though avoidance of math and science courses does not occur until high school, when such courses are no longer mandatory, the math filter subtly begins to sift girls out when they are in the middle grades. Summing up the themes that appear and reappear in the findings of several prominent researchers, Fox (1980) wrote:

> Although boys and girls do not differ with respect to achievement as measured by arithmetic tests in elementary school or on reported enjoyment of mathematics as a school subject, by the seventh grade striking sex differences emerge, most noticeably among the gifted; and by tenth or eleventh grade, when mathematics course-taking becomes optional, boys persist in the study of mathematics more than girls do. (p. 196)

By the high school level, then, negative attitudes toward math, and plans to exclude math and science electives, have been set, often irrevocably.

The middle-grade years are a uniquely favorable period for intervention, because girls' attitudes are still in flux, susceptible to judicious molding (Brush, 1980). Many young adolescent girls can still be convinced that math and science are enjoyable subjects, appropriate for females to study, and useful in attaining career goals. Once convinced, girls tend to take math and science courses, persevere, and achieve at a level not significantly lower than their male counterparts.

In order to be effective, intervention must be grounded in an understanding of the influences that cause girls to change their attitudes toward math and science. But it is important to remember that this change in attitude is, albeit important, only one of many highly significant changes that appear in the period known as early adolescence. "During adolescence," wrote Berryman (1983) in a report to the Rockefeller Foundation, "individuals are under simultaneous pressures to resolve sexual identities, form career preferences, and invest in training required to pursue their preferences. Scientific career interests and investments in high school mathematics are consistent with the development of masculine, but not feminine, identities" (p. 8).

Thus, occurring as it does in the midst of a maelstrom of physical, socio- emotional, and cognitive changes, the math filter phenomenon cannot be traced to a single cause. Rather, at the root of the matter is a complex combination of factors that do not operate equal on every individual (Kavrell & Petersen, in press). Biological and social influences have been cited by many researchers.

Researchers Disagree on Biological Factors

Within the past few years, some researchers have claimed biological difference between the sexes as a possible explanation for dissimilarities in math performance. A 1980 *Science* article by Benbow and Stanley seemed to claim a genetic basis for superior male achievement in math, based on the authors' study of mathematically precocious seventh graders. The article precipitated a flurry of controversy that received wide treatment in the popular press. Fellow researchers attacked Benbow and Stanley on every conceivable ground, criticizing the selection and recruitment of the sample population, the ages of the children studied, the perceived lack of attention to social influences, and the assumptions inherent in claiming that boys' and girls' mathematical training prior to seventh grade has been the same.

But the attack on Benbow and Stanley did not stem only from general outrage at their research design. Critics were eager to discredit the research because of the fear that if a solid biological foundation for sex-related differences in math achievement were found, all attempts to mitigate social influences would be abandoned as a lost cause. Benbow's often-quoted recommendation that "women would be better off accepting the differences" would seem to indicate that such fears were not entirely unfounded (Kolata, 1980, p. 1235).

Benbow and Stanley ended their article with this warning: "It...seems likely that putting one's faith in boy-versus-girl socialization processes as the only permissible explanation of the sex difference in mathematics is premature" (p. 1264). This basic warning, if viewed dispassionately, is correct. On the ground of academic freedom, no possible explanation of sex differences in math ability should be considered "impermissible." But even if genetic and therefore unchangeable differences in math ability between the sexes were eventually found, educators would still be duty-bound to help their male and female students reach their full potential. The emphasis, then, would be on developing new strategies to help girls master math and science.

A letter written by a group of scientists was among the many responses to Benbow and Stanley's research published in *Science* magazine. The six men and women concluded:

> Benbow and Stanley have not shown heritability or innate sex-linked differences in mathematical ability. Even if they had, it would tell us nothing about how to boost the observed performance of girls in mathematics. The real issue is whether or not one wants to see such a change. Attempts to attribute this social difference to a biological difference that would then legitimize the social differences are fallacious at best (Egelman, Alper, Leibowitz, Beckwith, Levine, & Leeds, 1981, p. 116).

Most evidence indicates that researchers have not been successful in identifying unchangeable genetic factors that may be responsible for sex-related differences in math ability. For example, the previously undisputed idea that boys had superior spatial ability (a trait thought to be related to math achievement) has been called into question in recent years (Armstrong, 1980). Further, researchers have repeatedly failed to find evidence linking superior spatial ability in boys to X-chromosome inheritance (Boles, 1980).

One promising developmental hypothesis, propounded and tested by Waber (1977), linked early and late maturation, independent of sex, to differences in verbal and spatial abilities. Based on tests of 80 elementary and high school youngsters, Waber found that "late maturers performed better than early maturers on spatial ability tests, but early maturers did not perform better than late maturers on verbal ability tests" (p. 34). Thus, according to Waber, "since females mature earlier than males, these data support the thesis that rate of physical maturation is an important determinant of the sex difference in spatial (but not verbal) ability" (p. 35). In other words, the difference in mathematical ability between the sexes may be linked to the fact that females mature faster, on the average, than males. "Maturational rate," Waber concluded, "has been shown to be a more powerful determinant of individual differences than sex itself on a set of psychological abilities on which the sexes have been found to differ" (p. 37).

Despite Waber's promising research, evidence for the influence of biologically based factors on math achievement has yielded sometimes inconclusive, sometimes contradictory results. As yet, the literature on rate of maturation and hormonal influence on cognitive functioning is still in its infancy.

Social Influences Shape Girls' Attitudes

Research on social influences on math achievement has been somewhat more conclusive. Fennema and Sherman (1977) found that stereotyping of math as a male domain inhibits girls' course taking, and, in turn, their achievement. Fox (1981) discussed girls' future expectations and their perceptions of math's future usefulness as major influences in their eventual success in math classes. Several others have described the impact of girls' expectations on eventual success or failure, describing female students' attitudes in terms of "math anxiety" (Tobias, 1976), "mathophobia" (Lazarus, 1974), and "math avoidance" (Sells, 1974). This is a complicated issue, primarily because a student can be influenced not only by her own expectations, but also by the expectations of her parents, counselors, teachers, and peers.

Early adolescence is a period during which self-perception comes not only from within, but as a reflection of the perception of others. Fennema and Sherman (1977) found that adolescent girls have less positive perceptions of their parents' confidence in their ability in mathematics than do boys. Fox (1977) reported that counselors seldom encouraged girls to pursue mathematics, and sometimes directly discouraged them. Ernest (1976) said that 41% of this sample of teachers believed that males did better in mathematics. Not one among his sample believed that females did better.

And, of course, there is the all-important peer influence. At no other stage in human development is there a stronger pressure to conform than during the period of early adolescence. And peer pressure to conform to appropriate sex-role stereotypes is particularly strong during this period. Thus, if girls perceive math and science course participation as inappropriate to their gender, they are less likely to enroll.

Most evidence on the expectations issue indicates that in the middle grades, girls begin to lose confidence in their ability as math achievers, even when there are no sex-related differences in mathematics performance. Confidence in learning mathematics is positively correlated with both math achievement and math course selection (Fennema & Sherman, 1977). When girls begin to perceive themselves as mathematically incompetent, they become mathematically incompetent. But such perceptions are not inevitable.

With Proper Help, Girls Achieve in Math and Science

Mounting evidence indicates that attempts to counter girls' negative attitudes toward math and science can be successful. In schools where conscious efforts have been made to encourage girls to believe in their ability, to persevere in math and science sequences, and to regard math and science as sex-appropriate pursuits, differences in math and science achievement scores are negligible or nonexistent (Brush, 1980; Paulsen & Johnson, 1983).

The elimination or narrowing of achievement discrepancies by sex occurs in some schools, not in others (Fennema, 1981). Moreover, in the schools with no differences in boys' and girls' math and science achievement, someone is likely to have made a conscious effort to counteract negative influences.

Fennema (1980) has identified a number of specific ways that teachers can counter the pressures on girls to avoid mathematics. First, teachers can eliminate negative references to women as learners or doers of mathematics in their own conversations and remarks. Second, since more girls than boys report that they are anxious about mathematics, teachers can encourage potentially anxious students in order to alleviate the pressure of mathophobia. Third, teachers can overtly discourage their students from stereotyping mathematics as a male domain. And fourth, teachers can make certain that students are aware of the importance of mathematics competence in keeping career options open.

"For middle school girls," Brush (1980) reported, "changes in the degree to which they see mathematics as useful for their future lives . . . predict changes in course plans" (p. 98). In other words, it is important that educators intervene to impress girls with the pragmatic benefits that will result from the study of mathematics. Brush suggested several other intervention techniques, including creating a "relaxed atmosphere in mathematics classrooms"; devising "special programs for parents, teachers, and guidance counselors on the usefulness of mathematics"; and introducing new, nontraditional mathematics courses, such as "teaching algebra through the medium of social science experiments," into the curriculum (pp. 99-101).

The critical factor for these intervention techniques, as well as for the programs that are described in the resource list at the end of this review, is timing. Girls must be reached in the middle grades, early enough to prevent the operation of the math and science filters. Once girls have opted out of the math sequence and are two years behind their male classmates in math experience, efforts to recruit women into math and science fields are doomed to failure.

Armstrong (1980) stated the case quite simply in the report of NIE's two-year "Women in Mathematics" study:

> Participation in mathematics is mostly an issue of awareness. If parents, teachers and counselors understand and transmit to students the necessity of taking mathematics to keep their options open and if they have the same high expectations for women as they have for men, then a basis for equal opportunity in scientific and technical fields will exist—for men and women. (p. 31)

In a dramatically expanding technological world, the pool of mathematically and scientifically talented individuals must also expand. If women are to be a part of that pool, attentiveness in the middle grades to girls' attitudes, achievement, and course selection is essential.

Resources

Books and Articles

Encouraging Girls in Mathematics: The Problem and the Solution, by Lorelei R. Brush, describes a longitudinal study that tracked female students' opinions about mathematics, their decisions regarding advanced mathematics courses, and their career plans at the 6th, 9th, and 12th grade levels. The author suggests ways to change girls' attitudes toward mathematics. *Available from: Abt Books, Cambridge, MA (1980)*

Overcoming Math Anxiety, by Shelia Tobias, discusses some of the reasons why people fear learning math, such as intimidation, misunderstanding, and missed opportunities. She also provides a math anxiety scale and includes information about programs for math-anxious people. *Available from: Houghton Mifflin, Boston, MA. (1978)*

Piaget and Middle School Mathematics, by William Juraschek, applies Piaget's theory of cognitive development to middle school mathematics. The typical middle school class contains some students operating at the concrete stage, some at the formal stage, and some in transition between the two cognitive stages. Juraschek argues that the traditional teaching style of teacher lecture followed by student seatwork is ineffective, given variations in early adolescent cognitive development. Activity-based teaching is recommended, and teaching suggestions are provided. *Published in: School Science and Mathematics, 1983, 83, 4-13.*

Teachers and Sex Bias in Mathematics, by Elizabeth Fennema, presents research indicating that teachers' behaviors differentially influence males and females in mathematics education. Fennema also suggests ways to reduce sexism in mathematics education. *Published in: Mathematics Teacher, 1980, 73, 169-173.*

What Research Says: Teaching Science to Middle School Students is an article in three parts by Patricia E. Blosser. Part I discusses the cognitive, physical, social, and emotional characteristics of young adolescents and the need of middle school science teachers to respond to these developmental characteristics. Part II presents several curriculum approaches. The final article addresses the characteristics of middle school science teachers, both personal and professional, and describes a sample teacher preparation program. *Published in: School Science and Mathematics, 1983, 83, 516-523; 609-615; 694-702.*

Curricula and Instructional Materials

Corrections: Women and Work and Skills for Good Jobs is a multimedia program for middle school students that addresses questions such as

"Why do men and women go to work?" and "How is monetary value placed on a job?" The program also is aimed at dispelling stereotyping of "male" and "female" occupations. It includes a leader's guide; a gamebook containing lessons, exercises, and activities; and "Women at Work," a sound filmstrip and script. *Available from: EDC/WEEA (see box).*

EQUALS promotes the participation of students and adults in mathematics courses and encourages their interest in math-based fields of study and work. It encompasses programs for teachers, counselors, administrators, parents, and scientific and technical workers. It also provides materials and models to improve student attitudes toward mathematics using a variety of instructional techniques. Consulting services include curriculum assessment, evaluation, and technical assistance. *Use EQUALS to Promote the Participation of Women in Mathematics* describes the teacher education program, provides methods and materials for educators, and includes annotated bibliographies on problem solving in mathematics and sex-fair counseling and instruction.

SPACES (Solving Problems of Access to Careers in Engineering and Science) is a collection of 32 math and career classroom activities designed for use in grades 4 through 10. The activities help students develop problem-solving skills while learning about mathematics-based fields of study and work. *Math for Girls and Other Problem Solvers*, by Diane Downie, Twila Slesnick, and Jean Kerr Stenmark, contains activities for children aged 7 to 13 on logic, spatial visualization, strategies, creative thinking, career awareness, and estimating and observing. The ideas presented are useful for classroom teachers, for parents and children, and for young people themselves. In addition, this book is organized so that teachers can easily identify curriculum objectives.

Family Math, a parent-child project, includes classes, materials and activities to use at home, and career information. *I'm Madly in Love with Electricity and Other Comments About Their Work in Science and Engineering* is a career booklet for young people curious about the sciences that includes a list of resource people and organizations. *Contact: Nancy Kreinberg, Director, EQUALS, Lawrence Hall of Science, University of California, Berkeley, CA 94720.*

Expanding Your Horizons in Science and Mathematics: A Handbook for Planners, developed by the Math/Science Network, describes how to plan, conduct, and evaluate conferences that aim to stimulate female involvement in math and science classes. The appendix includes six workshop activities plus useful forms, samples, quizzes, etc. *Available from: EDC/WEEA (see box).*

Guide to Nonsexist Teaching Activities (K-12), by the Northwest Regional Educational Laboratory, Center for Sex Equity, contains brief annotated descriptions of lesson plans, audiovisual materials, bibliographies and resource guides, games, course outlines, and periodicals that are equitable and sex-fair. The areas covered are general awareness, fine arts, counseling and career guidance, health and physical education, math and science, social studies, and sex equity organizations. Grade levels are provided for most of the entries, the majority of which are for the junior and senior high grades. There are six appendices, including a glossary, a model for developing nonsexist lesson plans, a lesson plan activity sheet, and guidelines for developing nonbiased instruction. Although designed primarily with the classroom teacher in mind, curriculum and media specialists, librarians, coaches, administrators, and school district Title IX coordinators should find it extremely useful. *Available from Oryx Press, Phoenix, AZ. (1983)*

The Math-Science Connection: Educating Young Women for Today is a 16mm film developed by the Math/Science Network for parents, educators, and other concerned adults. It discusses the causes and consequences of the rate of female participation in school math programs and reviews four model programs that stimulate and encourage girls and women to learn and enjoy mathematics. (Color, 18 minutes; also available as ¾" videocassette.) *Available from: EDC/WEEA (see box). (1979)*

MATHCO, developed by the Southwest Center for Human Relations Studies at the University of Oklahoma, Norman, helps girls in the middle grades feel competent about their ability to work with numbers, thus reducing anxiety. Program materials include posters, sound filmstrips, an inservice manual, and teachers guides. *Available from: EDC/WEEA (see box).*

Multiplying Options and Subtracting Bias, developed by Elizabeth Fennema, is an intervention program to eliminate sexism from mathematics education. It consists of four workshops for students, teachers, counselors, and parents. Each workshop is built around a 30-minute videotape narrated by Marlo Thomas. A facilitator's guide provides overhead transparencies, handout masters, and step-by-step instructions on conducting the workshops. (Four, 30-minute color sound videotapes [¾" videocassette, ½" open reel, VHS or Beta].) *Available for purchase from: National Council of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091. Available for rent from: Women and Mathematics Education, c/o Education Department, George Mason University, 4400 University Dr., Fairfax, VA 22030. (1981)*

Nothing But Options is an award-winning film produced by the Math/Science Network for those many teenage women who do not see how math confidence and competence can be useful in their present and future lives. Five articulate young career women, all of whom are down-to-earth, ambitious, and successful, offer information and encouragement. (17½ minute videotape; ¾", ½" VHS or Beta.) *Available from: Math/Science Network, c/o Mills College, Oakland, CA 94613. (1984)*

Sandra, Zella, Dee, and Claire, a film developed by the Math/Science Network, is intended for junior and senior high school women and men who might be interested in a career in science. An astronomer, a laser physicist, a veterinarian, and a mechanical engineer are interviewed on the job. They discuss their work, career opportunities, preparations for these and other fields, and the choices they made in their personal lives. (19 minutes, color, 16mm or ¾" videocassette.) *Available from: EDC/WEEA (see box).*

Spatial Encounters, developed at the Institute for Applied Research Services at the University of New Mexico, is a manual that contains exercises showing how various visual-spatial skills have applications in the real world of scientific occupations. The exercises, which include math, geography, acoustics, landscape architecture, and telecommunications, have clear statements of objectives and easy directions for classroom and self-directed use. *Available from: EDC/WEEA (see box).*

Women in Science is a series of videotapes that feature women with science careers, including dentistry, engineering, chemistry, computer science, geology, biomedical science, and physics/astronomy. The videotapes explore how and why the women chose the careers and the decisions they made regarding marriage and children. There also is a videotape that portrays the dilemma of a talented young woman interested in science and mathematics but discouraged from pursuing her interest by parents, peers, teachers, and counselors. Although the target audience is high school students, seventh and eighth graders would benefit from the videotapes as part of a career guidance/counseling program. (30 minutes each; color; ¾" U-matic; ½" VHS or Beta; User's Guide.) *Available for purchase or rent from: Women in Science Videotape Series, B322 School of Dentistry, University of Michigan, Ann Arbor, MI 48109. (1983)*

Women, Math, and Science: A Resource Manual, by the Center for Sex Equity in Schools at the University of Michigan, is an annotated resource list with sections on research literature; concerns of professional women in mathematics and science; programs and techniques for promoting sex equity; and organizations. Among the programs described are MATHCO and COMETS. *Available from: Dr. Charles D. Moddy, Sr., Director, Center for Sex Equity in Schools, 1046 School of Education, University of Michigan, Ann Arbor, MI 48109 (1982).*

Advocacy Resources

The **Association for Women in Mathematics** encourages women to enter and be active in mathematics and related careers, and promotes equal opportunity and equal treatment of women in the mathematical community. It has a Speaker's Bureau of women available to speak to a wide variety of audiences, including elementary and junior high school students. Members receive a bimonthly newsletter. *Address: Wellesley College, Wellesley, MA 02181.*

The **Math/Science Network** is a nonprofit corporation of over 1,200 scientists, educators, parents, community leaders, and business and industrial workers who work to promote women's participation in mathematics and science and to encourage women to enter nontraditional occupations. The Network acts as a planning resource, providing ideas, speakers, and materials, for organizations such as the National Science Foundation, the National Council of Teachers of Mathematics, and Girl Scouts of America. Network programs include "Expanding Your Horizons" conferences for secondary school young women held on college campuses throughout the U.S. (A number of Math/Science Network products are listed in previous sections of this resource list.) *Address: Math/Science Resource Center, c/o Mills College, Oakland, CA 94613.*

The **National Council of Teachers of Mathematics** is devoted to the interests of mathematics teachers and teacher educators. An information clearinghouse and resource center offers access to current materials. Publications include: *Activities for Junior High School and Middle School Mathematics; Arithmetic Teacher* and *Mathematics Teacher* (journals); and *Perspectives on Women and Mathematics. Address: 1906 Association Dr., Reston, VA 22091.*

PEER, the Project on Equal Education Rights of the NOW Legal Defense and Education Fund, works to end school practices, policies, and attitudes that limit children's choices or keep them from learning the skills they will need. Created in 1974, PEER helps parents and local community groups by offering advice, information, and in-depth training on how to initiate changes in schools. PEER publishes a number of titles, including *Equal Education Alert*, a news bulletin with up-to-date information on educational equity developments in Congress, federal agencies, and the courts, and the *Michigan PEER Math Study*, a report that analyzes math education in 113 Michigan public high schools and makes concrete recommendations for providing students with appropriate math preparation for today's high-tech society. *Address: 1413 K St. NW, 9th Floor, Washington, DC 20005.*

Women and Mathematics is a secondary school lectureship program developed by the Mathematical Association of America, which was concerned that women's inadequate preparation in mathematics would severely limit their career opportunities. The Women and Mathematics program sends women from a broad range of fields to talk to students about their careers and the importance of mathematics to their careers. The 200 speakers that are available also give presentations at career days, faculty workshops, and professional meetings, as well as to elementary school students and parent groups. *Address: % Department of Mathematics, University of Michigan, Flint, MI 48503.*

Women and Mathematics Education is an active networking organization that emphasizes the need for elementary and secondary programs to reverse female avoidance of mathematics. The organization also works to effect change within the mathematics education community and keeps its membership informed about issues affecting women in mathematics. Members receive a quarterly newsletter, an information brochure, and a resource list. *Address: % Education Department, George Mason University, Fairfax, VA 22030.*

Resources to Watch For

Challenge of the Unknown, a project of the American Association for the Advancement of Science, will focus on using mathematical problem solving in situations of interest to students in the middle grades. A series of films, microcomputer programs, and teacher notes is planned. *Contact: AAAS, 1515 Massachusetts Ave. NW, Washington, DC 20005.*

The **Computer Equity Training Project** wants to increase girls' computer use at the middle-grade level by developing school-based strategies to encourage girls to make greater voluntary use of computers. A fully tested manual for middle school personnel will be published in early Fall 1985. *Contact: Jo Suchat Sanders, Project Director, Computer Equity Training, Women's Action Alliance, Inc., 370 Lexington Ave., New York, NY 10017.*

The **Science Resources for Schools Project**, directed by the American Association for the Advancement of Science, will prepare school science packets designed to link schools with the scientific community. Prototypes, including information, ideas, and resource materials, are being used in junior high schools in Colorado, North Carolina, and Ohio. The project's purpose is to give all students, regardless of their career aspirations, a strong foundation in mathematics and science. *Contact: AAAS, 1515 Massachusetts Ave. NW, Washington, DC 20005.*

Women in Mathematics: Balancing the Equation, edited by Susan F. Chipman, Lorelei R. Brush, and Donna M. Wilson, represents the culmination of a series of research projects, funded by the National Institute of Education, that provide the background for designing effective educational programs to encourage women to enroll in mathematics beyond minimal school requirements. The expected date of publication is August 1984. *Available from: Lawrence Erlbaum Associates, Hillsdale, NJ.*

Resources compiled by Sue Rosenzweig.

EDC/WEEA Publishing Center

In 1974, Congress passed the Women's Equity Educational Act (WEEA) to promote educational equity for girls and women. The U.S. Department of Education has funded a variety of WEEA projects to develop materials and programs. The Education Development Center (EDC), in collaboration with the Center for Research on Women at Wellesley College, reviews, publishes, and distributes the material funded under WEEA. The EDC/WEEA Publishing Center is located at **55 Chapel St., Newton, MA 02160.** Phone: (800) 225-3088 (toll-free); (617) 969-7100 within Massachusetts.

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